

Well-posedness of higher order abstract Cauchy problems

Dissertation
der Mathematischen Fakultät
der Eberhard-Karls-Universität zu Tübingen

zur Erlangung des Grades eines Doktors
der Naturwissenschaften

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FRANK NEUBRANDER
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0. INTRODUCTION

Of concern is the abstract Cauchy problem

$$\begin{aligned} (\text{ACP}_{n+1}) \quad & u^{(n+1)}(t) - Au^{(n)}(t) - \sum_{k=1}^n B_k u^{(n-k)}(t) = 0 \\ & u^{(j)}(0) = x_j \quad \text{for } 0 \leq j \leq n, \end{aligned}$$

where A is a linear operator on a Banach space E with dense domain $D(A)$ and non empty resolvent set $\rho(A)$, and $B_1, B_2, \dots, B_n$ are closed linear operators with $D(A) \subset D(B_k)$ for $1 \leq k \leq n$.

A function $v(\cdot) \in C^{n+1}(\mathbb{R}^+, E)$ is said to be a solution of (ACP_{n+1}) if $v^{(n-k)}(t) \in D(B_k)$ for every $t \geq 0$, $0 \leq k \leq n$ ($B_0 := A$) and (ACP_{n+1}) is satisfied.

If (ACP_{n+1}) has a solution for every initial value $\bar{x} := (x_0, x_1, \dots, x_n) \in D$, where D is a dense subset of E^{n+1} , then (ACP_{n+1}) is said to be solvable for a dense set of initial data.

Concerning (ACP_{n+1}) the following four questions will be studied in this dissertation:

- (1) Is (ACP_{n+1}) solvable for a dense set of initial data?
- (2) Does it only have one solution for every initial value?

- (3) Can we find an explicit representation for these solutions?
- (4) How do the solutions depend on their initial data ?

The Cauchy problem (ACP_{n+1}) is called well-posed if it has unique solutions $u(\cdot)$ for a dense set of initial data which depend continuously on the initial data in the sense that $\lim_{n \rightarrow \infty} u_n^{(j)}(0) = 0$ for $0 \leq j \leq n$ implies $\lim_{n \rightarrow \infty} u_n(t) = 0$ uniformly on compact subsets of $\mathbb{R}^+$.

After introducing some notations in section 1 we will recall in section 2 some results on

$$(ACP_1) \quad u'(t) - Au(t) = 0, \quad u(0) = x,$$

which was introduced as an abstract Cauchy problem for differential equations in Banach space by E.Hille in 1952 [26]. It was shown by R.S.Phillips in 1954 [40] that existence and uniqueness of solutions for every initial data in the domain of A is equivalent to A being the generator of a strongly continuous semigroup on E . In this case it follows from elementary semigroup theory that (ACP_1) is well-posed. In 1967 H.O.Fattorini [14] proved that the class of semigroup generators coincides with the class of operators such that (ACP_1) is well-posed. S.G.Krein [29] and R.Beals [7] pointed out that the class of operators for which (ACP_1) has unique solutions for

some dense set of initial data is much larger than the class of semigroup generators. But, and this may explain that such operators have not been investigated systematically up to now, in these cases, by the result of Fattorini, (ACP_1) may not be well-posed. However, as we will prove in section 2, in many cases there is still a nice continuous dependence of the solutions from their initial data. At the end of the second section two new characterizations of a semigroup generator are presented. In section 3 these characterizations yield a condition on A such that (ACP_{n+1}) is solvable for a dense set of initial data. Our main result is the following:

Theorem

Let A be a densely defined linear operator and let $B_1, B_2, \dots, B_n$ be closed linear operators on a Banach space E with $D(A) \subset D(B_k)$ for $1 \leq k \leq n$. The following statements are equivalent:

- (i) A is the generator of a strongly continuous semigroup.
- (ii) A has non empty resolvent set and for every initial value $(x_0, x_1, \dots, x_n) \in (D(A))^{n+1}$ the Cauchy problem (ACP_{n+1}) has a unique solution $u(\cdot)$ in $C^{n+1}(\mathbb{R}^+, E) \cap C^n(\mathbb{R}^+, [D(A)])$ where $[D(A)]$ denotes the Banach space $D(A)$ endowed with the graph-norm $|x|_g := ||x|| + ||Ax||$.

For $n = 0$ this theorem is exactly the Phillips-characterization of a semigroup generator. Therefore it may be regarded as a generalization of Phillips' theorem to higher order abstract Cauchy problems.

In section 4 explicit expressions for the solutions in terms of the operators $A, B_1, B_2, \dots, B_n$ are given. The methods used there depend heavily on the inversion formula for the vector-valued Laplace transform.

In the appendix we will give an elementary proof and a more general version of the inversion formula of the vector-valued Laplace transform than those given in the hitherto existing literature.

In section 5 two well-posedness results are proved under additional assumptions on the operators $B_1, B_2, \dots, B_n$ which are frequently fulfilled in applications. The results of the sections 2, 3, 4 and 5 are then used in section 6 to discuss the solvability of the Cauchy problem for

$$u^{(2)}(t) - Au^{(1)}(t) - Bu(t) = f(t, u(t)) .$$

Among others it is shown how this damped nonlinear second order equation is related to the first order equation

$$u^{(1)}(t) = Au(t) + \int_0^t (Bu(s) + f(s, u(s))) ds .$$

Section 6 is divided into two parts. In the first one the equations are discussed under the assumption that A is an unbounded semigroup generator and that B is a closed

operator with $D(A) \subset D(B)$. As example, for an arbitrary semigroup generator A and every initial value $(x, y) \in (D(A))^2$, we construct the unique solution of the inhomogeneous damped second order abstract Cauchy problem

$$\begin{aligned} u^{(2)}(t) - (aA+bI)u^{(1)}(t) - (cA+dI)u(t) &= f(t), \\ u(0) = x, \quad u^{(1)}(0) = y, \quad a > 0, \quad b, c, d \in \mathbb{C}. \end{aligned}$$

Finally, in the second part of section 6 well-posedness of the damped second order equation is studied when A is a bounded and B a densely defined linear operator such that $\rho(B) \neq \emptyset$ and $AR(\lambda, B) = R(\lambda, B)A$. A unified treatment and some extensions of the results of Fattorini [14], Sova [46] and Da Prato - Giusti [9] are given.

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1. NOTATION

Let E be a Banach space with norm $||\cdot||$. $\sigma(A)$ ($p\sigma(A)$) denotes the spectrum (point spectrum) and $\rho(A)$ ($R(\lambda, A)$) the resolvent set (the resolvent) of a linear operator A with domain $D(A) \subset E$ and range in E . If A is a closed operator, then $[D(A^n)]$ denotes the Banach space

$$(D(A^n), |x|_n := ||x|| + ||Ax|| + \dots + ||A^n x||).$$

By $C^n(I, E)$ we denote the set of all n -times continuously differentiable functions, by $C(I, E)$ the set of all continuous functions from an interval I into E .

For every $m \in \mathbb{N}$, $|f|_m$ is defined as one of the following seminorms:

$$(i) \quad |f|_m := \sup_{t \in [0, m]} ||f(t)|| + \sup_{t \in [0, m]} ||f^{(1)}(t)|| + \dots + \sup_{t \in [0, m]} ||f^{(n)}(t)|| \\ \text{if } f \in C^n(\mathbb{R}^+, E),$$

$$(ii) \quad |f|_m := \sup_{t \in [0, m]} ||f(t)|| + \sup_{t \in [1/m, m]} ||f^{(1)}(t)|| \\ \text{if } f \in C^1((0, \infty), E) \cap C(\mathbb{R}^+, E).$$

$M_n, M_{1,0}$ are defined as the Fréchet-spaces

$$(C^n(\mathbb{R}^+, E), |\cdot|_m, m \in \mathbb{N}),$$

$$(C^1((0, \infty), E) \cap C(\mathbb{R}^+, E), |\cdot|_m, m \in \mathbb{N}).$$

For the elementary properties of strongly continuous semigroups we refer to [8], [20], [21], [27], [29], [39].

2. WELL-POSEDNESS OF FIRST ORDER ABSTRACT

CAUCHY PROBLEMS

We begin with a short summary of semigroup theory.

A family of bounded linear operators $\{T(t), t \geq 0\}$ on a Banach space E is called a strongly continuous semigroup $(T(t))$ if:

- (i) $T(0) = I$,
- (ii) $T(t+s) = T(t)T(s)$ for $t, s \geq 0$,
- (iii) $t \rightarrow T(t)x$ is continuous for every $x \in E$.

The generator A of the semigroup $(T(t))$ is defined by

$$Ax = \lim_{t \rightarrow 0+} \frac{1}{t}(T(t)x - x) = T'(0+)x$$

whenever the limit exists. The domain of A , denoted by $D(A)$, is the set of all $x \in E$ for which this limit exists. A is a densely defined, closed linear operator with non empty resolvent set $\rho(A)$.

For every strongly continuous semigroup $(T(t))$ there exist constants $w \in \mathbb{R}$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{wt}.$$

Every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > w$ is an element of the resolvent set of A and for such elements of $\mathbb{C}$ the resolvent $R(\lambda, A)$ of A is given as the Laplace transform of the semigroup $(T(t))$, i.e.

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt \text{ for every } x \in E$$

and every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > w$.

A linear operator A is the generator of a strongly continuous semigroup if and only if

- (i) A is densely defined,
- (ii) there exist $w \in \mathbb{R}$ and $M \geq 1$ such that every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > w$ is an element of $\rho(A)$ and $\|R(\lambda, A)^n\| \leq M(\operatorname{Re} \lambda - w)^{-n}$ for every $n \in \mathbb{N}$.

This is the famous Hille-Yosida characterization of a semigroup generator.

If A is the generator of a strongly continuous semigroup $(T(t))$, then, for every $x \in D(A)$, $t \mapsto T(t)x$ is the unique solution of the first order abstract Cauchy problem

$$(ACP_1) \quad u'(t) = Au(t), \quad u(0) = x.$$

Hille [25], Phillips [40] and Fattorini [14] gave different definitions of well-posedness of (ACP_1) . They showed

unique solution $u(\cdot) : \mathbb{R}^+ \rightarrow D(A)$ in $C^1(\mathbb{R}^+, E)$,
(well-posedness in the sense of Phillips).

- (iii) $\rho(A) \neq \emptyset$, for every $x \in D(A)$, (ACP_1) has a unique solution $u(\cdot) : \mathbb{R}^+ \rightarrow D(A)$ in $C^1((0, \infty), E) \cap C(\mathbb{R}^+, E)$ and there are positive real numbers a, M such that for every $t \in [0, a]$ and every solution $u(\cdot)$ of (ACP_1) we have

$$||u(t)|| \leq M ||u(0)||.$$

- (iv) A is closed and there exists a dense subset D such that for every $x \in D$, (ACP_1) has a solution $u(\cdot) : \mathbb{R}^+ \rightarrow D(A)$ in $C^1(\mathbb{R}^+, E)$. For every sequence of solutions with $u_n(0) \rightarrow 0$ we have $u_n(\cdot) \rightarrow 0$ uniformly on compacts of $\mathbb{R}^+$.

- (v) A is closed and there exists a dense subset D such that for every $x \in D$, (ACP_1) has a solution $u(\cdot) : \mathbb{R}^+ \rightarrow D(A)$ in $C^1((0, \infty), E) \cap C(\mathbb{R}^+, E)$. For every sequence $(u_n(\cdot))$ of solutions with $u_n(0) \rightarrow 0$ we have $u_n(\cdot) \rightarrow 0$ uniformly on compacts of $\mathbb{R}^+$, (well-posedness in the sense of Fattorini).

- (vi) A is closed and there exists a dense subset D such that for every $x \in D$, (ACP_1) has a solution $u(\cdot) : \mathbb{R}^+ \rightarrow D(A)$ in $C^1((0, \infty), E) \cap C(\mathbb{R}^+, E)$.

There are constants M, w such that for every solution $u(\cdot)$ and every $t \in \mathbb{R}^+$,

$$||u(t)|| \leq M e^{wt} ||u(0)||,$$

(well-posedness in the sense of Hille).

- (vii) A is closed and there exists a dense subset D such that for every $x \in D$, (ACP_1) has a solution $u(\cdot) : \mathbb{R}^+ \rightarrow D(A)$ in $C^1((0, \infty), E) \cap C(\mathbb{R}^+, E)$. There is a locally bounded function $f(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for every solution $u(\cdot)$ and every $t \in \mathbb{R}^+$,

$$||u(t)|| \leq f(t) ||u(0)||.$$

□

Before continuing, we make a few comments on the relation between semigroup generators and the Cauchy problem (ACP_1) . First we look at the definition of well-posedness in the sense of Phillips (statement (ii) of the previous theorem). If one defines (ACP_1) to be well-posed if

- (ii*) $\rho(A) \neq \emptyset$ and for every $x \in D(A)$, (ACP_1) has a unique solution $u(\cdot) : \mathbb{R}^+ \rightarrow D(A)$ in $C^1((0, \infty), E) \cap C(\mathbb{R}^+, E)$,

then this definition of well-posedness leads to so-called $(0, A)$ -semigroups which were studied in the book of Hille - Phillips [27]. It is shown there that the assumption that A generates a strongly continuous semigroup is not needed

in order to obtain well-posedness of (ACP_1) in the sense of (ii*).

We now vary the set of initial data (and not the space containing the solutions), i.e. (ACP_1) is defined - for the moment - to be well-posed, if there is a $k \geq 2$ such that

(ii_k) $\rho(A) \neq \emptyset$ and for every $x \in D(A^k)$, (ACP_1) has a unique solution $u(\cdot) : \mathbb{R}^+ \rightarrow D(A)$ in $C^1(\mathbb{R}^+, E)$.

It is shown by S.G. Krein [29, p.32] and more generally by R. Beals [7] that if A is a densely defined linear operator for which the resolvent exists in a right half-plane and satisfies the inequality

$$\|R(\lambda, A)\| \leq M(1 + |\lambda|)^{k-3},$$

then (ACP_1) is well-posed in the sense of (ii_k).

There is a large difference between the growth-condition on the resolvent stated above and the Hille-Yosida condition which estimates the growth for all powers of the resolvent. In this context we should mention a result of W. Arendt [1] :

Every generator of a positive semigroup (for a characterization see [2]) on a Banach lattice has positive resolvent. But the converse does not hold in general (for counterexamples see [3], [5] or [6]). However, if A

is a densely defined operator with positive resolvent on a Banach lattice, then (ACP_1) is well-posed in the sense of (ii_2) .

It is clear that if (ACP_1) is well-posed in the sense of (ii_k) , then, in general, not all solutions vary uniformly on compacts with their initial data in the sense of theorem 1 (iv). If so, then, by theorem 1, A would always be a semigroup generator.

However, there is still a fine relation between the variation of the initial data and their solutions as we will see in the following proposition.

Proposition 2 Let (ACP_1) be well-posed in the sense of (ii_k) and let $x_n(\cdot)$ be a sequence of solutions of (ACP_1) such that $x_n(0) \rightarrow 0$. Then the $(k-1)$ -times integrated solutions $f_n(\cdot)$,

$$f_n(t) := \int_0^t \frac{1}{(k-2)!} (t-s)^{k-2} x_n(s) ds ,$$

converge to zero uniformly on compacts of $\mathbb{R}^+$.

Proof: Let H be the operator from the Banach space $[D(A^k)]$ into the Fréchet-space M_k defined by

$$Hx := t \rightarrow \int_0^t \frac{1}{(k-2)!} (t-s)^{k-2} x(s) ds$$

where $x(\cdot)$ is the unique solution of (ACP_1) with initial value x in $D(A^k)$. H is an everywhere defined

closed linear operator. Therefore H is bounded.

Now let $x_n(\cdot)$ be a sequence of solutions of (ACP_1) such that $x_n := x_n(0) \rightarrow 0$ in E .

Define $u_n(\cdot) := R(\lambda, A)^k x_n(\cdot)$. Then $u_n(0)$ converges to zero in $[D(A^k)]$, $u_n(t) \in D(A^k)$ for every $t \in \mathbb{R}^+$ and $u_n(\cdot)$ is a solution of (ACP_1) .

$Hu_n(0)$ converges to zero in M_k which implies

- a) $\int_0^t \frac{1}{j!} (t-s)^j u_n(s) ds \rightarrow 0$ uniformly on compacts for $0 \leq j \leq k-2$,
- b) $u_n(\cdot) \rightarrow 0$ uniformly on compacts and
- c) $u_n'(\cdot) = Au_n(\cdot) \rightarrow 0$ uniformly on compacts.

It is easy to see that $u_n(\cdot)$ is k -times continuously differentiable, $u_n^{(j)}(\cdot) = A^j u_n(\cdot)$ for $1 \leq j \leq k$ and that $u_n^{(j)}(0) = A^j R(\lambda, A)^k x_n(0) \rightarrow 0$ for $1 \leq j \leq k$.

From $x_n(t) = (\lambda - A)^k u_n(t)$

$$= \sum_{j=0}^k (-1)^j \binom{k}{j} \lambda^{k-j} A^j u_n(t)$$

$$= \sum_{j=0}^k (-1)^j \binom{k}{j} \lambda^{k-j} u_n^{(j)}(t)$$

we obtain

$$f_n(t) = \int_0^t \frac{1}{(k-2)!} (t-s)^{k-2} x_n(s) ds$$

$$= \sum_{j=0}^k (-1)^j \binom{k}{j} \lambda^{k-j} \int_0^t \frac{1}{(k-2)!} (t-s)^{k-2} u_n^{(j)}(s) ds.$$

$$\begin{aligned}
\text{But} \quad & \int_0^t \frac{1}{(k-2)!} (t-s)^{k-2} u_n^{(j)}(s) \, ds \\
&= \int_0^t \frac{1}{(k-2-j)!} (t-s)^{k-2-j} u_n(s) \, ds \\
&\quad - \sum_{i=0}^{j-1} \frac{1}{(k-1-j+i)!} t^{k-1-j+i} u_n^{(i)}(0)
\end{aligned}$$

for $1 \leq j \leq k-2$,

$$\begin{aligned}
& \int_0^t \frac{1}{(k-2)!} (t-s)^{k-2} u_n^{(k-1)}(s) \, ds \\
&= u_n(t) - \sum_{i=0}^{k-2} \frac{1}{i!} t^i u_n^{(i)}(0)
\end{aligned}$$

$$\begin{aligned}
\text{and} \quad & \int_0^t \frac{1}{(k-2)!} (t-s)^{k-2} u_n^{(k)}(s) \, ds \\
&= u_n'(t) - \sum_{i=0}^{k-2} \frac{1}{i!} t^i u_n^{(i+1)}(0) .
\end{aligned}$$

Therefore $f_n(\cdot) \rightarrow 0$ uniformly on compacts of $\mathbb{R}^+$.

□

The remarks above show that the definition of well-posedness explaining best the relation between semigroup generators and (ACP_1) is the one given by Fattorini:

If one wants solutions of (ACP_1) for a dense set of initial data which vary uniformly on compacts with their initial data, then one has to show that A is the generator of a strongly continuous semigroup;

if one only wants that the solutions vary in a weaker sense with their initial data, the class of appropriate operators A becomes larger.

We conclude the discussion of theorem 1 with the following problem:

Characterize those closed operators A such that (ACP_1) is well-posed in the sense of (ii_k) .

For a first discussion of this problem we refer to a forthcoming paper of W. Arendt and the author.

The second theorem of this section is a modified version of the characterization of a semigroup generator by well-posedness of (ACP_1) in the sense of Phillips.

Theorem 3 Let A be a linear operator on a Banach space E such that $D(A^{n+1})$ is dense in E for some $n \in \mathbb{N} \cup \{0\}$. Then the following statements are equivalent:

- (i) A is the generator of a strongly continuous semigroup.
- (ii) $\rho(A) \neq \emptyset$ and, for every $x \in D(A^{n+1})$, (ACP_1) has a unique solution $u(\cdot) : \mathbb{R}^+ \rightarrow D(A)$ in $C^{n+1}(\mathbb{R}^+, E)$.

Proof:

(i) $\rightarrow$ (ii) follows from elementary semigroup theory.

We show: (ii) implies that (ACP_1) is well-posed in the sense of Fattorini. A is densely defined with non empty resolvent set, hence closed.

For every $x \in D := D(A^{n+1})$ (ACP_1) has a unique solution $u : \mathbb{R}^+ \rightarrow D(A)$ in $C^{n+1}(\mathbb{R}^+, E) \subset C^1(\mathbb{R}^+, E)$.

We define a linear operator $K : [D(A^{n+1})] \rightarrow M_{n+1}$ by $x \rightarrow Kx := u(\cdot)$, where $u(\cdot)$ is the unique solution of (ACP_1) . K is a closed, everywhere defined and hence continuous linear operator.

Now let $u_k(\cdot)$ be a sequence of solutions of (ACP_1) such that $u_k(0) \rightarrow 0$ for $k \rightarrow \infty$. We will show that $u_k(\cdot) \rightarrow 0$ uniformly on compacts of $\mathbb{R}^+$ (well-posedness in the sense of Fattorini).

Define $v_k(\cdot) := R(\lambda, A)^{n+1} u_k(\cdot)$.

Then $v_k(\cdot) : \mathbb{R}^+ \rightarrow D(A^{n+1})$

and $A^j v_k(0) \rightarrow 0$ for $k \rightarrow \infty$ and every $0 \leq j \leq n+1$.

Hence $v_k(0) \rightarrow 0$ in $[D(A^{n+1})]$.

Now we show by induction that $v_k^{(j-1)}(\cdot)$ is differentiable and that $v_k^{(j)}(\cdot) = A^j v_k(\cdot)$ for every $1 \leq j \leq n+1$.

The statement is true for $j = 1$. Suppose it is true for every $1 \leq m \leq j$ where $1 \leq j \leq n$. Then it is true for $j+1$ as can be seen by

$$(\lambda - A)^j v_k(\cdot) = R(\lambda, A)^{n+1-j} u_k(\cdot) \quad \text{or}$$

$$v_k^{(j)}(\cdot) = A^j v_k(\cdot)$$

$$= (-1)^j [R(\lambda, A)^{n+1-j} u_k(\cdot) - \sum_{r=1}^j \binom{j}{r} \lambda^r (-1)^{j-r} v_k^{(j-r)}(\cdot)] .$$

Therefore $v_k^{(j)}(\cdot)$ is continuously differentiable and

$$\begin{aligned} v_k^{(j+1)}(\cdot) &= (-1)^j [R(\lambda, A)^{n+1-j} u_k(\cdot) - \sum_{r=1}^j \binom{j}{r} \lambda^r (-1)^{j-r} v_k^{(j-r+1)}(\cdot)] \\ &= A [(-1)^j [R(\lambda, A)^{n+1-j} u_k(\cdot) - \sum_{r=1}^j \binom{j}{r} \lambda^r (-1)^{j-r} v_k^{(j-r)}(\cdot)]] \\ &= A^{j+1} v_k(\cdot) . \end{aligned}$$

We have seen that $v_k(\cdot)$ is a solution of (ACP_1) in $C^{n+1}(\mathbb{R}^+, E)$ and by the continuity of K we obtain

$v_k(\cdot) = K v_k(0) + 0$ in M_{n+1} which implies that

$v_k^{(j)}(\cdot) \rightarrow 0$ uniformly on compacts of $\mathbb{R}^+$ for every

$0 \leq j \leq n+1$. But

$$\begin{aligned} u_k(\cdot) &= (\lambda - A)^{n+1} v_k(\cdot) \\ &= \sum_{r=0}^{n+1} \binom{n+1}{r} (\lambda)^r (-1)^{n+1-r} v_k^{n+1-r}(\cdot) \end{aligned}$$

and therefore $u_k(\cdot) \rightarrow 0$ uniformly on compacts of $\mathbb{R}^+$.

□

In the subsequent sections we will give sufficient conditions for solvability and well-posedness of

$$\begin{aligned} (\text{ANP}_{n+1}) \quad & u^{(n+1)}(t) - Au^{(n)}(t) - \sum_{k=1}^n B_k u^{(n-k)}(t) = 0 \\ & u^{(j)}(0) = x_j \quad \text{for } 0 \leq j \leq n \end{aligned}$$

where A is a densely defined linear operator on a Banach space E with non empty resolvent set $\rho(A)$ and $B_1, B_2, \dots, B_n$ are closed linear operators with $D(A) \subset D(B_k)$ for $1 \leq k \leq n$.

The results there depend heavily on the following characterization of a semigroup generator. In the proof we make use of the Phillips-perturbation-theorem (see, for example, [41]) which says that if A is a semigroup generator and B is a bounded linear operator, then $A+B$ is a semigroup generator too.

Theorem 4

Let A be a densely defined linear operator and let $B_1, B_2, \dots, B_n$ be closed linear operators on E with $D(A) \subset D(B_k)$ for $1 \leq k \leq n$. Then the following statements are equivalent:

- (i) A is the generator of a strongly continuous semigroup on E .

(ii)

$$\Delta := \begin{bmatrix} A & I & 0 & \cdot & \cdot & 0 \\ B_1 & 0 & I & \cdot & \cdot & 0 \\ B_2 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & I \\ B_n & 0 & 0 & \cdot & \cdot & 0 \end{bmatrix}$$

is the

generator of a strongly continuous semigroup on E^{n+1} .

First Proof:

a) (i) $\rightarrow$ (ii). Let $\lambda \in \rho(A)$.

Define $A_0 := (-\lambda + A)$. Then A_0 is the generator of a strongly continuous semigroup $(T(t))$, $0 \in \rho(A_0)$ and $D(A_0) = D(A)$. Define

$$\Psi := \begin{bmatrix} \lambda I & I & 0 & \cdot & \cdot & 0 \\ 0 & 0 & I & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & I \\ 0 & 0 & 0 & \cdot & \cdot & 0 \end{bmatrix}$$

and

$$\Delta_0 := \begin{bmatrix} A_0 & 0 & 0 & \cdot & \cdot & 0 \\ B_1 & 0 & 0 & \cdot & \cdot & 0 \\ B_2 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ B_n & 0 & 0 & \cdot & \cdot & 0 \end{bmatrix}.$$

Then $\Delta := \Delta_0 + \Psi$. We will see that Δ_0 is the generator of a strongly continuous semigroup on E^{n+1} . Hence, by the Phillips Perturbation Theorem, Δ is the generator of a strongly continuous semigroup.

From the closedness of the operators B_k ($1 \leq k \leq n$) and from $D(A) \subset D(B_k)$ we obtain that $B_k R(0, A_0)$ is a continuous linear operator on E ($1 \leq k \leq n$).

For every $t \in \mathbb{R}^+$ define a bounded linear operator $\Gamma(t)$ on E^{n+1} by

$$\Gamma(t) := \begin{bmatrix} T(t) & 0 & 0 & \cdot & \cdot & 0 \\ B_1 R(0, A_0) (I - T(t)) & I & 0 & \cdot & \cdot & 0 \\ B_2 R(0, A_0) (I - T(t)) & 0 & I & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ B_n R(0, A_0) (I - T(t)) & 0 & 0 & \cdot & \cdot & I \end{bmatrix}.$$

By simple matrix calculations one shows that $(\Gamma(t))$ is a strongly continuous semigroup on E^{n+1} with generator Δ_0 .

b) (ii) $\rightarrow$ (i). If Δ is the generator of a strongly continuous semigroup, then the same is true for Δ_0 . Therefore there is a $w > 0$ such that every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > w$ is an element of $\rho(\Delta_0)$ and

$$\|R(\lambda, \Delta_0)^n\| \leq M(\operatorname{Re} \lambda - w)^{-n} \quad \text{for every } n \in \mathbb{N}.$$

This implies that for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > w > 0$ there are bounded linear operators X_{ij} on E such that for the matrix (X_{ij}) :

$$(X_{ij})(\lambda - \Delta_0) = (\lambda - \Delta_0)(X_{ij}) = I.$$

In particular we obtain

$$(\lambda - A_0)X_{11} = I,$$

$$X_{11}(\lambda - A_0) - X_{12}B_1 - X_{13}B_2 - \dots - X_{1n+1}B_n = 0,$$

$$\lambda X_{12} = \lambda X_{13} = \dots = \lambda X_{1n+1} = 0.$$

Therefore $(\lambda - A_0)X_{11} = X_{11}(\lambda - A_0) = I$ and every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > w$ is an element of $\rho(A_0)$. Define

$$R_\lambda := \begin{bmatrix} R(\lambda, A_0) & 0 & 0 & \dots & 0 \\ \frac{1}{\lambda} B_1 R(\lambda, A_0) & \frac{1}{\lambda} I & 0 & \dots & 0 \\ \frac{1}{\lambda} B_2 R(\lambda, A_0) & 0 & \frac{1}{\lambda} I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\lambda} B_n R(\lambda, A_0) & 0 & 0 & \dots & \frac{1}{\lambda} I \end{bmatrix}.$$

It can be easily seen that $R_\lambda = R(\lambda, \Delta_0)$ and that $R(\lambda, \Delta_0)^n$ is of the form

$$R(\lambda, \Delta_0)^n = \begin{bmatrix} R(\lambda, A_0)^n & 0 & 0 & \cdot & \cdot & 0 \\ X_1 & X_2 & \cdot & \cdot & \cdot & X_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Z_1 & Z_2 & \cdot & \cdot & \cdot & Z_n \end{bmatrix}$$

where $X_1, \dots, Z_i$ are any bounded linear operators on E . Hence for the densely defined linear A_0 with non empty resolvent set it follows that

$$||R(\lambda, A_0)^n|| \leq ||R(\lambda, \Delta_0)^n|| \leq M(\operatorname{Re} \lambda - w)^{-n}$$

for every $n \in \mathbb{N}$ and every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > w$.

By the Hille-Yosida characterization A_0 and hence A is the generator of a strongly continuous semigroup.

Second Proof:

Define

$$\Delta_1 := \begin{bmatrix} A & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \end{bmatrix}, \quad \Delta_2 := \begin{bmatrix} 0 & 0 & \cdot & 0 \\ B_1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ B_n & 0 & \cdot & 0 \end{bmatrix},$$

$$\Psi := \begin{bmatrix} 0 & I & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_\lambda := \begin{bmatrix} R(\lambda, A) & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda} I & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \frac{1}{\lambda} I \end{bmatrix}.$$

If A is a semigroup generator on E , then Δ_1 is a semigroup generator on E^{n+1} , $D(\Delta_1) = D(A) \times E^n$ and for $\lambda \in R(\lambda, A) \setminus \{0\}$ the resolvent $R(\lambda, \Delta_1)$ is given by R_λ . Δ_2 is a continuous linear operator from the Banach space $[D(A)] \times E^n$ into E^{n+1} . This can be seen by the following considerations: Let $x_k \rightarrow 0$ in $[D(A)] \times E^n$. Then $(\lambda - \Delta_1)x_k \rightarrow 0$ in E^{n+1} . But $\Delta_2 R(\lambda, \Delta_1)$ is a continuous operator on E^{n+1} and therefore $\Delta_2 x_k \rightarrow 0$ in E^{n+1} . Hence, by $\Delta_1 \Delta_2 = 0$ and the perturbation theorem of Desch and Schappacher [11], $\Delta_1 + \Delta_2 + \Psi = \Delta$ is a semigroup generator on E^{n+1} .

Conversely, if $\Delta_1 + \Delta_2$ is a semigroup generator on E^{n+1} ($D(\Delta_1 + \Delta_2) = D(A) \times E^n$), then the resolvent set of A is non empty and $R(\lambda, \Delta_1 + \Delta_2)$ is given by

$$\begin{bmatrix} R(\lambda, A_0) & 0 & 0 & \cdot & \cdot & 0 \\ \frac{1}{\lambda} B_1 R(\lambda, A_0) & \frac{1}{\lambda} I & 0 & \cdot & \cdot & 0 \\ \frac{1}{\lambda} B_2 R(\lambda, A_0) & 0 & \frac{1}{\lambda} I & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{\lambda} B_n R(\lambda, A_0) & 0 & 0 & \cdot & \cdot & \frac{1}{\lambda} I \end{bmatrix}.$$

From the continuity of $\Delta_2 R(\lambda, \Delta_1 + \Delta_2)$ on E^{n+1} it

follows that $-\Delta_2$ is continuous from $[D(A)]x E^n$ into E^{n+1} and hence, by $\Delta_1 \Delta_2 = 0$, from $[D(A)]x E^n$ into $[D(A)]x E^n$. Therefore, by the same perturbation theorem, Δ_1 and hence A is a semigroup generator.

□

If A is the generator of a compact or holomorphic semigroup so is $A + B$ for every bounded B (see, for example, [41]). In the proof of theorem 3 we have seen that the semigroup generated by

$$\Delta_0 := \begin{bmatrix} A_0 & 0 & 0 & \cdot & \cdot & 0 \\ B_1 & 0 & 0 & \cdot & \cdot & 0 \\ B_2 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ B_n & 0 & 0 & \cdot & \cdot & 0 \end{bmatrix}$$

is given by

$$\Gamma(t) := \begin{bmatrix} T(t) & 0 & 0 & \cdot & \cdot & 0 \\ B_1 R(0, A_0) (I - T(t)) & I & 0 & \cdot & \cdot & 0 \\ B_2 R(0, A_0) (I - T(t)) & 0 & I & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ B_n R(0, A_0) (I - T(t)) & 0 & 0 & \cdot & \cdot & I \end{bmatrix} .$$

From this and the fact that Δ is a semigroup generator

on E^{n+1} if and only if A is a semigroup generator on E we obtain the following corollary.

Corollary 6 Let A be a densely defined linear operator and let $B_1, B_2, \dots, B_n$ be closed linear operators on E with $D(A) \subset D(B_k)$ for $1 \leq k \leq n$. Then the following statements are equivalent:

- (i) A is the generator of a compact or holomorphic semigroup on E .
- (ii) A is the generator of a compact or holomorphic semigroup on E^{n+1} .

3. SOLVABILITY OF HIGHER ORDER ABSTRACT CAUCHY PROBLEMS

In [16] H.O. Fattorini considered the Cauchy problem

$$(\#) \quad u^{(2)}(t) - Au^{(1)}(t) - Bu(t) = 0$$

where A and B are densely defined linear operators on a Banach space E .

He studied properties of the solutions under the assumption that $(\#)$ is solvable for a dense subset of initial data such that the solutions depend continuously on its initial data, i.e. there exists locally bounded functions $K_0(\cdot)$, $K_1(\cdot)$ from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that

$$\|u(t)\| \leq K_0(t) \|u(0)\| + K_1(t) \|u^{(1)}(0)\|$$

for every solution $u(\cdot)$ of $(\#)$.

Further he wrote:

"We shall not be concerned here with the problem of finding conditions on the coefficients A, B in order that the Cauchy problem for $(\#)$ be well posed in some sense or another..."

This section deals with the problem of finding conditions on $A, B_1, B_2, \dots, B_n$ such that the Cauchy problem for

$$(*) \quad u^{(n+1)}(t) - Au^{(n)}(t) - \sum_{k=1}^n B_k u^{(n-k)}(t) = 0$$

is solvable for a dense set of initial data.

We remark that Phillips' definition of well-posedness for (ACP_1) and statement (ii) in the following theorem agree when $n = 0$. Therefore one may regard the following result as a generalization of the Phillips- characterization of a semigroup generator to higher order abstract Cauchy problems.

For a discussion of uniqueness of the solutions in $C^{n+1}(\mathbb{R}^+, E)$ and their continuous dependence from the initial data, see section 5.

Theorem 7

Let A be a densely defined linear operator and let $B_1, B_2, \dots, B_n$ be closed linear operators on a Banach space E with $D(A) \subset D(B_k)$ for $1 \leq k \leq n$. Consider the following Cauchy problem:

$$(ACP_{n+1}) \quad \begin{aligned} u^{(n+1)}(t) - Au^{(n)}(t) - \sum_{k=1}^n B_k u^{(n-k)}(t) &= 0 \\ u^{(j)}(0) &= x_j, \quad 0 \leq j \leq n. \end{aligned}$$

Then the following statements are equivalent:

- (i) A is the generator of a strongly continuous semigroup.
- (ii) A has non empty resolvent set and for every initial value $(x_0, x_1, \dots, x_n) \in (D(A))^{n+1}$ the Cauchy problem (ACP_{n+1}) has a unique solution $u(\cdot)$ in $C^{n+1}(\mathbb{R}^+, E) \cap C^n(\mathbb{R}^+, [D(A)])$.
- (iii) A has non empty resolvent set and for every initial value $(x_0, x_1, \dots, x_n) \in (D(A))^{n+1}$ the Cauchy problem (ACP_{n+1}) has a solution $u(\cdot)$ in $C^{n+1}(\mathbb{R}^+, E)$ such that
- a) $u^{(n-k)}(\cdot) : \mathbb{R}^+ \rightarrow D(B_k)$ for $0 \leq k \leq n$,
 $(B_0 := A)$,
- b) $B_k u^{(n-k)}(\cdot)$ are continuous for $1 \leq k \leq n$.

The solution is unique among all solutions satisfying a) and b).

Proof:

a) (i) $\rightarrow$ (ii). The proof of this implication is divided into two main parts. In the first one (lemma 7.1) we prove that the operator

$$\Psi := \begin{bmatrix} I & 0 & 0 & \cdot & \cdot & 0 \\ -A & I & 0 & \cdot & \cdot & 0 \\ -B_1 & -A & I & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & I & 0 \\ -B_{n-1} & -B_{n-2} & \cdot & -B_1 & -A & I \end{bmatrix}$$

maps $D(A)^{n+1}$ onto $D(\Delta^{n+1})$ where Δ is defined by

$$\Delta := \begin{bmatrix} A & I & 0 & \cdot & \cdot & 0 \\ B_1 & 0 & I & \cdot & \cdot & 0 \\ B_2 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & I \\ B_n & 0 & 0 & \cdot & \cdot & 0 \end{bmatrix}.$$

By theorem 4, Δ is the generator of a strongly continuous semigroup on E^{n+1} . Hence, by theorem 3, the Cauchy problem

$$u'(t) = \Delta u(t), \quad u(0) = \bar{\psi} \bar{x}$$

has for every $\bar{x} \in D(A)^{n+1}$ a unique solution $u(\cdot)$ in $C^{n+1}(\mathbb{R}^+, E^{n+1})$. In the second part, beginning with lemma 7.2, we show that the first coordinate of this solution is a solution of (ACP_{n+1}) in $C^{n+1}(\mathbb{R}^+, E) \cap C^n(\mathbb{R}^+, [D(A)])$ with initial value $\bar{x} := (x_0, x_1, \dots, x_n) \in D(A)^{n+1}$.

Lemma 7.1. Ψ is a bijection from $D(A)^{n+1}$ onto $D(\Delta^{n+1})$.

Proof: An induction shows that $\Delta^k \Psi$ ($1 \leq k \leq n$) is given by

$$\begin{bmatrix} 0 & y_k \\ z_k & 0 \end{bmatrix} \quad \text{where}$$

$$z_k := \begin{bmatrix} B_n & B_{n-1} & \cdot & \cdot & B_{n-k+1} \\ 0 & B_n & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & B_n & B_{n-1} \\ 0 & 0 & \cdot & 0 & B_n \end{bmatrix},$$

$$y_k := \begin{bmatrix} I & 0 & 0 & \cdot & \cdot & 0 \\ -A & I & 0 & & & 0 \\ -B_1 & -A & I & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ -B_{n-1-k} & \cdot & \cdot & -B_1 & -A & I \end{bmatrix},$$

$$y_{n-1} := \begin{bmatrix} I & 0 \\ -A & I \end{bmatrix}, \quad y_n := I.$$

From this we obtain $\bar{\psi} \bar{x} \in D(\Delta^{n+1})$ for every $\bar{x} = (x_0, x_1, \dots, x_n) \in D(A)^{n+1}$. It is easy to see that $\bar{\psi}$ is an injective map. By another induction one shows that $D(\Delta^k)$ is rekursively given by

$$D(\Delta) = \{ (y_0, y_1, \dots, y_n) \in E^{n+1} : x_0 := y_0 \in D(A) \} ,$$

$$D(\Delta^{k+1}) = \{ (y_0, y_1, \dots, y_n) \in E^{n+1} : x_j \in D(A) \text{ for } 0 \leq j \leq k-1 \text{ and } x_k := \sum_{i+j=k-1} B_i x_j + y_k \in D(A) \text{ } (B_0 := A) \} .$$

Now let $(y_0, y_1, \dots, y_n) \in D(\Delta^{n+1})$. Define $x_0 := y_0$ and

$$x_k := \sum_{i+j=k-1} B_i x_j + y_k \quad (B_0 := A) .$$

Then $(x_0, x_1, \dots, x_n) \in D(A)^{n+1}$ and $\bar{\psi}(x_0, x_1, \dots, x_n) = (-\sum_{i+j=k-1} B_i x_j + x_k)_{0 \leq k \leq n} = (y_0, y_1, \dots, y_n)$. $\square$

Lemma 7.2

Let A be a densely defined linear operator with non empty resolvent set. The following statements are equivalent:

$$(i) \quad u(\cdot) \in C^n(\mathbb{R}^+, E) \quad , \quad u(\cdot) : \mathbb{R}^+ \rightarrow D(A) \quad , \\ Au(\cdot) \in C^n(\mathbb{R}^+, E) .$$

$$(ii) \quad u(\cdot) \in C^n(\mathbb{R}^+, [D(A)]) .$$

$$(iii) \quad u(\cdot) \in C^n(\mathbb{R}^+, E) \quad , \quad u(\cdot) : \mathbb{R}^+ \rightarrow D(A) \quad \text{and} \\ Bu(\cdot) \in C^n(\mathbb{R}^+, E) \quad \text{for every closed operator } B \\ \text{with } D(A) \subset D(B) .$$

Proof: The implication (i) $\rightarrow$ (ii) follows from the definition of the Banach space $[D(A)]$. To prove the implication (ii) $\rightarrow$ (iii) let $u(\cdot)$ be an element of $C^n(\mathbb{R}^+, [D(A)])$. From the closedness of B , $D(A) \subset D(B)$ and $\rho(A) \neq \emptyset$ we conclude that B is a continuous operator from $[D(A)]$ into E . Looking at

$$\mathbb{R}^+ \xrightarrow{u(\cdot) \in C^n} [D(A)] \xrightarrow{B} E$$

we see that $Bu(\cdot) \in C^n(\mathbb{R}^+, E)$.

The implication (iii) $\rightarrow$ (i) is obvious. $\square$

Now let $(x_0, x_1, \dots, x_n) \in D(A)^{n+1}$ and define

$$\begin{aligned} (y_0, y_1, \dots, y_n) &:= \Psi(x_0, x_1, \dots, x_n) \\ &= (-\sum_{i+j=k-1} B_i x_j + x_k)_{0 \leq k \leq n} \in D(\Delta^{n+1}). \end{aligned}$$

Using theorem 3 and 4 we see that there exists a unique solution $u(\cdot) : \mathbb{R}^+ \rightarrow D(\Delta)$ in $C^{n+1}(\mathbb{R}^+, E^{n+1})$ of

$$u'(t) = \Delta u(t) \quad , \quad u(0) = (y_0, y_1, \dots, y_n).$$

This implies that there are functions $u_i(\cdot) \in C^{n+1}(\mathbb{R}^+, E)$,
 $1 \leq i \leq n+1$, with $u_1(\cdot) : \mathbb{R}^+ \rightarrow D(A)$,
 $(u_1(0), u_2(0), \dots, u_{n+1}(0)) = (y_0, y_1, \dots, y_n)$ and

$$\begin{aligned}
 & Au_1(t) + u_2(t) = u_1'(t) \\
 & B_1 u_1(t) + u_3(t) = u_2'(t) \\
 (1) \quad & \cdot \quad + \quad \cdot \quad = \quad \cdot \\
 & \cdot \quad + \quad \cdot \quad = \quad \cdot \\
 & B_{n-1} u_1(t) + u_{n+1}(t) = u_n'(t) \\
 & B_n u_1(t) = u_{n+1}'(t) .
 \end{aligned}$$

By the closedness of $A, B_1, \dots, B_n$ and the differentiability of the functions $u_i(\cdot)$ it is easy to see that

$$u_1^{(j)}(\cdot) : \mathbb{R}^+ \rightarrow D(A) \quad \text{and}$$

$$\frac{d}{dt} Au_1^{(j)}(\cdot) = Au_1^{(j+1)}(\cdot)$$

for $0 \leq j \leq n-1$. Hence, by lemma 7.2, $u_1(\cdot)$ is an element of $C^{n+1}(\mathbb{R}^+, E) \cap C^n(\mathbb{R}^+, [D(A)])$.

Now, looking at (1), we obtain

$$u_1^{(n+1)}(t) - Au_1^{(n)}(t) - \sum_{k=1}^n B_k u_1^{(n-k)}(t) = 0 \quad \text{and}$$

$$u_1^{(k)}(0) = Au_1^{(k-1)}(0) + \sum_{j=1}^{k-1} B_j u_1^{(k-1-j)}(0) + u_{k+1}(0)$$

for $1 \leq k \leq n$.

Therefore:

$$\begin{aligned}u_1^{(0)}(0) &= u_1(0) = y_0 = x_0, \\u_1^{(1)}(0) &= Au_1(0) + u_2(0) = Ax_0 + y_1 = x_1, \\u_1^{(2)}(0) &= Ax_1 + B_1x_0 + y_2 = \sum_{i+j=1} B_i x_j + y_2 = x_2\end{aligned}$$

and by induction we obtain

$$(u_1^{(0)}(0), u_1^{(1)}(0), \dots, u_1^{(n)}(0)) = (x_0, x_1, \dots, x_n).$$

We have shown: given $\bar{x} = (x_0, x_1, \dots, x_n) \in D(A)^{n+1}$, then the first coordinate $u_1(\cdot)$ of the solution of

$$u'(t) = \Delta u(t), \quad u(0) = \Psi \bar{x}$$

is a solution of (ACP_{n+1}) in $C^{n+1}(\mathbb{R}^+, E) \cap C^n(\mathbb{R}^+, [D(A)])$ with initial value $(x_0, x_1, \dots, x_n)$.

Suppose there is another solution $v(\cdot)$ of (ACP_{n+1}) in $C^{n+1}(\mathbb{R}^+, E) \cap C^n(\mathbb{R}^+, [D(A)])$ with initial value $(x_0, \dots, x_n)$.

Then $w(\cdot) := v(\cdot) - u_1(\cdot)$ is a solution of (ACP_{n+1}) in $C^{n+1}(\mathbb{R}^+, E) \cap C^n(\mathbb{R}^+, [D(A)])$ with

$$w^{(k)}(0) = 0 \quad \text{for } 0 \leq k \leq n.$$

Define $z(t) := (w_1(t), \dots, w_{n+1}(t))$

where $w_1(t) := w(t)$ and

$$w_k(t) := w^{(k-1)}(t) - Aw^{(k-2)}(t) - \sum_{j=1}^{k-2} B_j w^{(k-2-j)}(t).$$

The function $z(\cdot): \mathbb{R}^+ \rightarrow D(\Delta) = D(A) \times E^n$ is an element of $C^1(\mathbb{R}^+, E^{n+1})$ (see lemma 7.2 (iii)) with $z(0) = (0, 0, \dots, 0)$.

It is easy to see that

$$w'_k(t) = w_{k+1}(t) + B_{k-1}w(t), \quad 1 \leq k \leq n.$$

Hence

$$\begin{aligned} \Delta z(t) &= (Aw_1(t) + w'_1(t) - Aw_1(t), B_1w_1(t) + w'_2(t), \dots, \\ &\quad \dots, B_{n-1}w_1(t) + w'_{n+1}(t), B_nw_1(t)) \\ &= (w'_1(t), w'_2(t), \dots, w'_n(t), B_nw_1(t)) \end{aligned}$$

and

$$\begin{aligned} w'_{n+1}(t) &= w^{(n+1)}(t) - Aw^{(n)}(t) - \sum_{k=1}^{n-1} B_k w^{(n-k)}(t) \\ &= B_n w_1(t) \end{aligned}$$

(since $w_1(\cdot)$ is a solution of (ACP_{n+1})).

Therefore $\Delta z(t) = z'(t), z(0) = 0$.

But Δ is a semigroup generator and therefore $z(\cdot) = 0$, $w(\cdot) = 0$ or $u_1(\cdot) = v(\cdot)$.

b) The equivalence of statements (ii) and (iii) follows from the following considerations. Let $u(\cdot)$ be a solution of (ACP_{n+1}) in $C^{n+1}(\mathbb{R}^+, E) \cap C^n(\mathbb{R}^+, [D(A)])$. Then

$u^{(n-k)}(\cdot) : \mathbb{R}^+ \rightarrow D(A) \subset D(B_k)$ for $0 \leq k \leq n$ ($B_0 := A$)
and by lemma 7.2 we obtain the continuity of $B_k u^{(n-k)}(\cdot)$
for $1 \leq k \leq n$.

If $u(\cdot)$ is a solution of (ACP_{n+1}) with the properties
a) and b) of statement (iii), then

$$t \rightarrow Au^{(n)}(t) = u^{(n+1)}(t) - \sum_{k=1}^n B_k u^{(n-k)}(t)$$

is a continuous function. Therefore

$$u^{(n-1)}(t) = \int_0^t u^{(n)}(s) ds \in D(A) \text{ for every } t \geq 0 \text{ and}$$

$$t \rightarrow Au^{(n-1)}(t) = u^{(n)}(t) - \int_0^t \sum_{k=1}^n B_k u^{(n-k)}(s) ds$$

is continuously differentiable. By induction we obtain:

- 1) $u^{(j)}(t) \in D(A)$ for every $t \geq 0$ and every
 $0 \leq j \leq n$,
- 2) $Au^{(j)}(\cdot)$ are continuously differentiable for
 $0 \leq j \leq n-1$.

Hence $u(\cdot) \in C^{n+1}(\mathbb{R}^+, E)$, $u(\cdot) : \mathbb{R}^+ \rightarrow D(A)$ and
 $Au(\cdot) \in C^n(\mathbb{R}^+, E)$. Therefore, by lemma 7.2,
 $u(\cdot) \in C^{n+1}(\mathbb{R}^+, E) \cap C^n(\mathbb{R}^+, [D(A)])$.

(c) (ii) $\rightarrow$ (i) : Let $(y_0, y_1, \dots, y_n) \in D(\Delta^{n+1})$. Then,
by lemma 7.1, there is a unique $(x_0, x_1, \dots, x_n) \in D(A)^{n+1}$
with $\Psi(x_0, x_1, \dots, x_n) = (y_0, y_1, \dots, y_n)$ where $x_0 := y_0$
and $x_k := \sum_{i+j=k-1} B_i x_j + y_k$ ($B_0 := A$).

Let $u(\cdot) \in C^{n+1}(\mathbb{R}^+, E) \cap C^n(\mathbb{R}^+, [D(A)])$ be the unique solution of (ACP_{n+1}) with initial value $(x_0, x_1, \dots, x_n)$. We will show that for every $(y_0, y_1, \dots, y_n) \in D(\Delta^{n+1})$ the Cauchy problem $u'(t) = \Delta u(t)$, $u(0) = (y_0, y_1, \dots, y_n)$ has a unique solution in $C^{n+1}(\mathbb{R}^+, E^{n+1})$. Then, by theorem 3 and 4, Δ and hence A is a semigroup generator.

Define $u_0(t) := u(t)$,

$$u_k(t) := \sum_{j=k}^n \int_0^t \frac{1}{(j-k)!} (t-s)^{j-k} B_j u_0(s) ds + \frac{1}{(j-k)!} t^{j-k} y_j$$

and $w(t) = (u_0(t), u_1(t), \dots, u_n(t))$.

Then $w(0) = (y_0, y_1, \dots, y_n)$ and, by lemma 7.2, $w(\cdot) \in C^{n+1}(\mathbb{R}^+, E^{n+1})$.

From the definition of the functions $u_k(\cdot)$ it follows that

$$\begin{aligned} B_1 u_0(t) + u_2(t) &= u_1'(t) \\ B_2 u_0(t) + u_3(t) &= u_2'(t) \\ &\cdot \quad + \quad \cdot \quad = \quad \cdot \\ &\cdot \quad + \quad \cdot \quad = \quad \cdot \\ B_{n-1} u_0(t) + u_n(t) &= u_{n-1}'(t) \\ B_n u_0(t) &= u_n'(t) \end{aligned}$$

Therefore, in order to show that $w(\cdot)$ is a solution of the differential equation $u'(t) = \Delta u(t)$ it suffices to prove the equation

$$\begin{aligned}
Av_0(t) + v_1(t) &= v_0'(t) \\
B_1 v_0(t) + v_2(t) &= v_1'(t) \\
&\vdots \\
&\vdots + \vdots = \vdots \\
B_{n-1} v_0(t) + v_n(t) &= v_{n-1}'(t)
\end{aligned}$$

we obtain $v(\cdot) = (v_0(\cdot), v_1(\cdot), \dots, v_n(\cdot)) = 0$. Therefore $w(\cdot) \in C^{n+1}(\mathbb{R}^+, E^{n+1})$ is the unique solution of the Cauchy problem $u'(t) = Au(t)$, $u(0) = (y_0, y_1, \dots, y_n) \in D(\Delta^{n+1})$.

□

Perhaps the most restrictive assumption in theorem 5 is $D(A) \subset D(B_k)$ for $1 \leq k \leq n$. If A is a bounded operator, then the statement of the theorem is well known (every B_k has to be bounded then). For bounded A in general nothing can be said about existence of solutions of (ACP_{n+1}) if one of the operators B_i is unbounded. In section 6 we will discuss such cases, more precisely we investigate equations

$$(\#) \quad u^{(2)}(t) - Au^{(1)}(t) - Bu(t) = 0$$

for bounded A and densely defined B such that the resolvent set of B is nonempty and $AR(\lambda, B) = R(\lambda, B)A$.

Whereas the case mentioned above leads to so-called cosine generators, another important case of the equation (#)

can be treated with semigroup theory only.

Sandefur [42], [43], [44] and Aviles and Sandefur [4], discussed iterated differential equations

$$(\#\#) \quad \prod_{i=1}^n \left(\frac{d}{dt} - A_i \right) w(t) = 0$$

where A_i are semigroup generators on a Banach space E such that

$D := \{ D(A_{i(1)} \cdots A_{i(m)}) ; i(j) \in \{1, \dots, n\}, m = 1, 2, \dots \}$ is dense in E . For $n = 2$ ($\#\#$) becomes

$$(\#\#\#) \quad w^{(2)}(t) - (A_1 + A_2)w^{(1)}(t) - A_1 A_2 w(t) = 0.$$

Sandefur proved well-posedness of ($\#\#$) for every $1 \leq k \leq n$ (along the lines of Fattorini) under the assumptions on the operators A_i mentioned above.

In the case ($\#\#\#$) $A_1 + A_2$ need not be a semigroup generator and nothing can be said about the relation between $D(A_1 + A_2)$ and $D(A_1 A_2)$.

For further results on higher order Cauchy problems and references to the literature we refer to [14], [15], [17], [18], [25], [31], [49] and a paper of Weiss [50] which was, together with [43], the starting point of our investigations.

4. EXPLICIT SOLUTIONS OF HIGHER ORDER

CAUCHY PROBLEMS

In this section we show a way how to come to an explicit expression for the solution of (ACP_{n+1}) . As proved in the previous section the solution of (ACP_{n+1}) is given by the first coordinate of the semigroup $\Gamma(t)$ generated by Δ as defined in theorem 4.

The first and main step to obtain this "first coordinate" is the formula for $R(\lambda, \Delta)$ given in the next proposition.

Proposition 8

Let $A, B_1, B_2, \dots, B_n$ be densely defined closed operators and let $D(A) \subset D(B_k)$ for $1 \leq k \leq n$. Define Δ as in theorem 4. Then the following statements are equivalent:

(i) $\lambda \in \rho(\Delta)$.

(ii) $R_\lambda := (\lambda^n(\lambda - A) - \lambda^{n-1}B_1 - \dots - B_n)^{-1}$ exists.

(Remark: in the scalar case R_λ corresponds to $\det(\lambda - \Delta)$.

If $\lambda \in \rho(\Delta)$ then $R(\lambda, \Delta)$ is given by

$$\begin{bmatrix} \lambda^n R_\lambda & \lambda^{n-1} R_\lambda & \lambda^{n-2} R_\lambda & \cdot & \cdot & R_\lambda \\ Y_1 R_\lambda & \lambda^{n-1} X_1 R_\lambda & \lambda^{n-2} X_1 R_\lambda & \cdot & \cdot & X_1 R_\lambda \\ \lambda Y_2 R_\lambda & Y_2 R_\lambda & \lambda^{n-2} X_2 R_\lambda & \cdot & \cdot & X_2 R_\lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda^{n-1} Y_n R_\lambda & \lambda^{n-2} Y_n R_\lambda & \cdot & Y_n R_\lambda & X_n R_\lambda \end{bmatrix}$$

where

$$X_1 := \lambda - A,$$

$$X_k := (\lambda^{k-1}(\lambda - A) - \lambda^{k-2}B_1 - \dots - B_{k-1}) \quad (k \geq 2)$$

and

$$Y_k := \sum_{j=k}^n \lambda^{n-k} B_k.$$

Proof:

(i) $\rightarrow$ (ii). Let $\lambda \in \rho(\Delta)$.

Then there are bounded operators X_{ij} on E such that

$$(X_{ij})(\lambda - \Delta) = (\lambda - \Delta)(X_{ij}) = I.$$

This leads to the following equations:

$$(1) \quad X_{1,n+1}(\lambda^n(\lambda - A) - \lambda^{n-1}B_1 - \dots - B_n) = I$$

$$\text{and } X_{1,k} = \lambda^{n+1-k} X_{1,n+1} \quad \text{for } 1 \leq k \leq n.$$

$$(2) \quad X_{i,1}(\lambda - A) - X_{i,2}B_1 - \dots - X_{i,n+1}B_n = 0$$

$$(2 \leq i \leq n+1),$$

$$X_{i,k} = \lambda^{n+1-k} X_{i,n+1} \quad \text{for } i \leq k \leq n+1$$

$$\text{and } x_{i,k} = \lambda^{i-1-k} (\lambda^{n+2-i} x_{i,n+1} - I) \text{ for } k \leq i-1 .$$

$$(3) \quad (\lambda - A)x_{1,1} - x_{2,1} = I$$

$$\text{and } (\lambda - A)x_{1,k} - x_{2,k} = 0 \text{ for } 2 \leq k \leq n+1 .$$

$$(4) \quad -B_1 x_{1,k} + \lambda x_{i+1,k} - x_{i+2,k} = \delta_{i+1,k} I$$

$$\text{for } 1 \leq i \leq n-1 \text{ and } 1 \leq k \leq n+1 .$$

$$(5) \quad -B_n x_{1,k} + \lambda x_{n+1,k} = \delta_{n+1,k} I \text{ for } 1 \leq k \leq n+1 .$$

Looking at (3) and (4) we obtain:

$$\begin{aligned} & (\lambda - A)x_{1,n+1} = x_{2,n+1} \\ & -B_1 x_{1,n+1} + \lambda x_{2,n+1} = x_{3,n+1} \\ & -B_2 x_{1,n+1} + \lambda x_{3,n+1} = x_{4,n+1} \\ & \quad \cdot \\ & \quad \cdot \\ & \quad \cdot \\ & -B_{n-1} x_{1,n+1} + \lambda x_{n,n+1} = x_{n+1,n+1} \\ & -B_n x_{1,n+1} + \lambda x_{n+1,n+1} = I \end{aligned}$$

From (6) we obtain

$$(7) \quad (\lambda^n (\lambda - A) - \lambda^{n-1} B_1 - \dots - B_n) x_{1,n+1} = I .$$

Therefore, by (1) and (7),

$$R_\lambda := X_{1,n+1} = (\lambda^n(\lambda-A) - \lambda^{n-1}B_1 - \dots - B_n)^{-1}$$

exists. By (6) we obtain expressions for $X_{k,n+1}$ for $2 \leq k \leq n+1$ and by (1), (2) and (3) for all $X_{i,k}$. This leads to the formula for $R(\lambda, \Delta)$ stated in the proposition.

(ii) $\rightarrow$ (i). If R_λ exists, then define $R(\lambda, \Delta)$ as above and show that $R(\lambda, \Delta)(\lambda - A) = (\lambda - A)R(\lambda, \Delta) = I$. We remark that no commutativity of the operators $R_\lambda, A, B_1, \dots, B_n$ is needed to prove the first equality.

□

Corollary 9

Let A be the generator of a strongly continuous semigroup and let $\{B_1, B_2, \dots, B_n\}$ be closed operators with $D(A) \subset D(B_k)$ for $1 \leq k \leq n$. Then there exists a $w > 0$ such that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > w$, $R_\lambda := (\lambda^n(\lambda-A) - \lambda^{n-1}B_1 - \dots - B_n)^{-1}$ exists. Let $p > w$ and define γ as the path $\{p+i\mathbb{R}\}$. Then a solution of (ACP_{n+1}) is given by

$$\frac{1}{2\pi i} \int_\gamma e^{\lambda t} \sum_{k=0}^n \lambda^{n-k} R_\lambda \left[\left(-\sum_{j=0}^{k-1} B_{k-1-j} x_j \right) + x_k \right] d\lambda$$

for every initial value $(x_0, x_1, \dots, x_n) \in (D(A))^{n+1}$.

Proof: Let $\bar{x} := (x_0, x_1, \dots, x_n) \in (D(A))^{n+1}$.

Define Ψ as in the proof of theorem 7, part a).

Then $\Psi(x_0, x_1, \dots, x_n) = (z_0, z_1, \dots, z_n)$

where $z_k = - \sum_{j=0}^{k-1} B_{k-1-j} x_j + x_k \quad (B_0 := A)$.

Define Δ as in theorem 4. Then, as shown in the proof of theorem 7, part a), a solution of (ACP_{n+1}) is given as the first coordinate of $\Gamma(t)\Psi\bar{x}$, where $\Gamma(t)$ is the semigroup generated by Δ . Using the inversion formula for the Laplace transform,

$$\text{i.e.} \quad \Gamma(t)\Psi\bar{x} = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} R(\lambda, \Delta) \Psi\bar{x} d\lambda$$

(see appendix, theorem A.5) and proposition 8 we obtain the expression for the solution of (ACP_{n+1}) stated in the corollary.

□

At the first glance the formula for the solution as given in corollary 9 seems to be rather difficult. However, there are a lot of examples in which an explicit expression for the solution can be easily determined by the formula above (see the examples in the sections 5 and 6).

5. WELL-POSEDNESS OF HIGER ORDER ABSTRACT CAUCHY PROBLEMS

The Cauchy problem (ACP_{n+1}) for

$$(\#) \quad u^{(n+1)}(t) - Au^{(n)}(t) - \sum_{k=1}^n B_k u^{(n-k)}(t) = 0$$

is called well-posed if

(i) (ACP_{n+1}) is solvable for a dense subset D of initial data $\bar{x} = (x_0, x_1, \dots, x_n)$ in E^{n+1} .

(ii) The solutions depend continuously on their initial data, i.e.:

Let $(v_k(\cdot))$ be a sequence of solutions of (ACP_{n+1}) such that $v_k^{(j)}(0) \rightarrow 0$ in E for $k \rightarrow \infty$ and $0 \leq j \leq n$. Then $v_k(\cdot) \rightarrow 0$ uniformly on compacts of $\mathbb{R}^+$.

We remark that assumption (ii) implies uniqueness of the solutions and we start our discussion of well-posedness with a fundamental lemma which will be of frequent use in the following (for a version of the lemma for $n = 1$, see section 6).

Lemma 10

Let A be the generator of a strongly continuous semigroup and let $\{B_1, B_2, \dots, B_n\}$ be

closed operators with $D(A) \subset D(B_k)$ for $1 \leq k \leq n$. Then there are strongly continuous families $(T_{i,j}(t))$ ($t \geq 0$, $0 \leq i, j \leq n$) of bounded linear operators such that the semigroup $\Gamma(t)$ generated by

$$\Lambda := \begin{bmatrix} A & I & 0 & \cdot & \cdot & 0 \\ B_1 & 0 & I & \cdot & \cdot & 0 \\ B_2 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & I \\ B_n & 0 & 0 & \cdot & \cdot & 0 \end{bmatrix}$$

is given by

$$\begin{bmatrix} T_{0,0}(t) & \cdot & \cdot & T_{0,n}(t) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ T_{h,0}(t) & \cdot & \cdot & T_{n,n}(t) \end{bmatrix}.$$

The families $(T_{i,j}(t))$ have the following properties:

- a) $T_{0,0}(t) : D(A) \rightarrow D(A)$ and $T_{0,j}(t) : E \rightarrow D(A)$ for $1 \leq j \leq n$.
- b) If $0 \leq i \leq n-1$ and $1 \leq j \leq n$, then $t \mapsto T_{i,j}(t)x$ is continuously differentiable for every $x \in E$ and (where necessary, define $B_0 := A$) $T'_{i,j}(t)x = T_{i,j-1}(t)x = B_i T_{0,j}(t)x + T_{i+1,j}(t)x$.

c) If $0 \leq i \leq n-1$ and $j = 0$, then $t \mapsto T_{i,0}(t)x$ is continuously differentiable for every $x \in D(A)$ and

$$\begin{aligned} T'_{i,0}(t)x &= \sum_{k=0}^n T_{i,k}(t)B_k x \\ &= B_i T_{0,0}(t)x + T_{i+1,0}(t)x \end{aligned}$$

d) If $i = n$ and $j = 0$, then $t \mapsto T_{n,0}(t)x$ is continuously differentiable for every $x \in D(A)$ and

$$\begin{aligned} T'_{n,0}(t)x &= \sum_{k=0}^n T_{n,k}(t)B_k x \\ &= B_n T_{0,0}(t)x \end{aligned}$$

e) If $i = n$ and $1 \leq j \leq n$, then $t \mapsto T_{n,j}(t)x$ is continuously differentiable for every $x \in E$ and

$$T'_{n,j}(t)x = T_{n,j-1}(t)x = B_n T_{0,j}(t)x$$

f) Let $\bar{x} := (x_0, x_1, \dots, x_n) \in (D(A))^{n+1}$. Then

$$\begin{aligned} v(t) &:= \sum_{k=0}^n T_{0,k}(t)[x_k - \sum_{j=0}^{k-1} B_{k-1-j}x_j] \\ &= \sum_{i=0}^{n-1} [T_{0,i}(t)x_i - \sum_{k=i+1}^n T_{0,k}(t)B_{k-1-i}x_i] \\ &\quad + T_{0,n}(t)x_n \end{aligned}$$

is a solution of (ACP_{n+1}) with initial value $\bar{x}$.

Proof:

The statements a) - e) follow from the facts that $D(\Delta)$ is given by $D(A)x E^n$,

$\Gamma(t) : D(\Delta) \rightarrow D(\Delta)$ and that $\frac{d}{dt}\Gamma(t)\bar{x} = \Delta\Gamma(t)\bar{x} = \Gamma(t)\Delta\bar{x}$ for every $\bar{x} \in D(\Delta)$. Statement f) follows from the proof of theorem 7.

□

Corollary 11

Let A be the generator of a strongly continuous semigroup and let $\{B_1, B_2, \dots, B_n\}$ be bounded linear operators. Then (ACP_{n+1}) is well-posed.

Proof:

Let $u(\cdot)$ be any solution of (ACP_{n+1}) with initial values $u^{(j)}(0) = x_j$. By the continuity of $B_k u^{(n-k)}(\cdot)$ ($1 \leq k \leq n$) we obtain by the same considerations as in the proof of theorem 7, part b), that $u^{(j)}(\cdot) : \mathbb{R}^+ \rightarrow D(A)$. Hence $x_j \in D(A)$. By theorem 7 (iii), we obtain uniqueness of the solutions. Therefore, by the previous lemma, part f), $u(\cdot)$ is given by

$$\sum_{k=0}^n T_{0,k}(t) [x_k - \sum_{j=0}^{k-1} B_{k-1-j} x_j].$$

From $||T_{i,j}(t)|| \leq ||\Gamma(t)|| \leq Me^{wt}$

we obtain continuous dependence of the solutions from the initial data and hence, by definition, well-posedness of (ACP_{n+1}) .

□

In order to explain the hypothesis of the following theorem we will consider for a moment the second order

equation

$$(\#) \quad u^{(2)}(t) - Au^{(1)}(t) - Bu(t) = 0 .$$

Examples are:

the strongly damped wave equation

$$u^{(2)}(t) - a\Delta u^{(1)}(t) - \Delta u(t) = 0 , \quad a > 0 ,$$

the linear strongly damped Klein-Gordon equation

$$u^{(2)}(t) - 2a\Delta u^{(1)}(t) - (\Delta + bI)u(t) = 0$$

and the vibrating beam equation

$$u^{(2)}(t) + 2(a\Delta^2 + b)u^{(1)}(t) + \Delta^2 u(t) = 0 .$$

We see that (#) frequently appears in the form

$$u^{(2)}(t) - (aA + bI)u^{(1)}(t) - (cA + dI)u(t) = 0 ,$$

$$a > 0 , \quad b, c, d \in \mathbb{C} .$$

For a discussion of this equation see page 80 . In this case $R_\lambda = \frac{1}{a\lambda + c} R(\frac{1}{a\lambda + c}(\lambda^2 - \lambda b - d), A)$ and hence, for $B := cA + dI$ we obtain $BR_\lambda = R_\lambda B$ on $D(A)$ which is the additional assumption in the following theorem.

Theorem 12

Let A be the generator of a strongly continuous semigroup and let $\{B_1, B_2, \dots, B_n\}$ be closed operators with

- (i) $D(A) \subset D(B_k)$ for $1 \leq k \leq n$.
- (ii) $B_k R_\lambda = R_\lambda B_k$ on $D(A)$ for $1 \leq k \leq n$ and $\lambda \in \{p+i\mathbb{R}\}$, $p > w$ (w defined as in corollary 9).

Then (ACP_{n+1}) is well-posed and every solution with initial value $(x_0, x_1, \dots, x_n)$ is given by

$$\sum_{k=0}^n T_{n-k,n}(t) x_k$$

where the operators $T_{n-k,n}(t)$ are defined as in lemma 10.

Proof: a) From $B_k R_\lambda = R_\lambda B_k$ on $D(A)$ for $1 \leq k \leq n$ it follows that $A R_\lambda = R_\lambda A$ on $D(A)$. Let $\lambda, \mu \in \{p+i\mathbb{R}\}$. For every $x \in E$ define $y := R_\lambda x \in D(A)$. Then

$$\begin{aligned} R_\lambda R_\mu x &= R_\lambda R_\mu (\lambda^n (\lambda - A)^{-n} - \lambda^{n-1} B_1 - \dots - B_n) y \\ &= R_\lambda (\lambda^n (\lambda - A)^{-n} - \lambda^{n-1} B_1 - \dots - B_n) R_\mu y = R_\mu R_\lambda x. \end{aligned}$$

Therefore $R_\lambda R_\mu x = R_\mu R_\lambda x$ for $\lambda, \mu \in \{p+i\mathbb{R}\}$.

b) As seen in section 4, the operators $T_{0,j}(t)$ ($0 \leq j \leq n$) are given as inverse Laplace transform of the functions $\lambda \rightarrow \lambda^{n-j} R_\lambda$, i.e.

$$T_{0,j}(t)x = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} \lambda^{n-j} R_\lambda x \, d\lambda$$

for every $x \in E$ (if $1 \leq j \leq n$) or for every $x \in D(A)$ (if $j = 0$). Therefore, by the closedness of the operators B_k ($0 \leq k \leq n$, $B_0 := A$), we obtain $T_{0,j}(t)B_k = B_k T_{0,j}(t)$ on $D(A)$ for $0 \leq j, k \leq n$.

c) By lemma 10, part f), a solution of (ACP_{n+1}) with initial value $(x_0, x_1, \dots, x_n) \in (D(A))^{n+1}$ is given by

$$\sum_{i=0}^{n-1} S_i(t)x_i + T_{0,n}(t)x_n,$$

$$\begin{aligned} \text{where } S_i(t)x &:= T_{0,i}(t)x - \sum_{k=i+1}^n T_{0,k}(t)B_{k-1-i}x \\ &= T_{0,i}(t)x - \sum_{k=i+1}^n B_{k-1-i}T_{0,k}(t)x \end{aligned}$$

for $x \in D(A)$. By lemma 10, part b), we obtain

$$\begin{aligned} S_i(t)x &:= T_{0,i}(t)x - \sum_{k=0}^{n-i-1} T_{k,k+i}(t)x \\ &\quad + \sum_{k=0}^{n-i-1} T_{k+1,k+1+i}(t)x \\ &= T_{n-i,n}(t)x. \end{aligned}$$

d) Let $u(\cdot)$ be a solution of (ACP_{n+1}) with $u^{(j)}(0) = 0$ for $0 \leq j \leq n$. Define $v(\cdot) := R_\lambda u(\cdot)$. Then $v(\cdot)$ is a solution of (ACP_{n+1}) with $v^{(j)}(0) = 0$ for $0 \leq j \leq n$ and $B_k v^{(n-k)}(\cdot) = B_k R_\lambda u^{(n-k)}(\cdot)$ are continuous functions. Hence, by theorem 7 (iii), $v(\cdot) = R_\lambda u(\cdot) = 0$ or $u(\cdot) = 0$. Therefore, every solution of (ACP_{n+1}) is unique.

e) Let $\bar{x} = (x_0, x_1, \dots, x_n) \in (D(A))^{n+1}$. Then the unique solution of (ACP_{n+1}) with initial value $\bar{x}$ is given by

$$\sum_{i=0}^n T_{n-i,n}(t) x_i.$$

Let $(v_k(\cdot))$ be a sequence of solutions of (ACP_{n+1}) such that $v_k^{(j)}(0) \rightarrow 0$ for $k \rightarrow \infty$ and $0 \leq j \leq n$. Define $u_k(\cdot) := R_\mu v_k(\cdot)$. Then $u_k(\cdot)$ is a solution of (ACP_{n+1}) with $u_k^{(j)}(0) \in D(A)$ for $0 \leq j \leq n$. Therefore

$$u_k(t) = R_\mu v_k(t) = \sum_{i=0}^n T_{n-i,n}(t) R_\mu v_k^{(i)}(0).$$

As seen in section 4, the operators $T_{k,n}(t)$ ($0 \leq k \leq n$) are given as inverse Laplace transforms of the functions $\lambda \rightarrow X_k R_\lambda$ where $X_0 = I$, $X_1 = (\lambda - A)$, $X_k = (\lambda^{k-1}(\lambda - A) - \lambda^{k-2}B_1 - \dots - B_{k-1})$ ($k \geq 2$), i.e.:

$$T_{k,n}(t)x = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} X_k R_\lambda x \, d\lambda \quad \text{for every } x \in E.$$

Now let $\mu \in \gamma = \{p+i\mathbb{R}\}$. Then, as proved in part a),

$$\begin{aligned} T_{k,n}(t) R_{\mu} x &= \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} X_k R_{\lambda} R_{\mu} x \, d\lambda \\ &= R_{\mu} \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} X_k R_{\lambda} x \, d\lambda \\ &= R_{\mu} T_{k,n}(t) x \end{aligned}$$

for every $x \in E$.

Therefore

$$u_k(t) = R_{\mu} v_k(t) = R_{\mu} \left(\sum_{i=0}^n T_{n-i,n}(t) v_k^{(i)}(0) \right)$$

$$\text{or} \quad v_k(t) = \sum_{i=0}^n T_{n-i,n}(t) v_k^{(i)}(0).$$

Hence, by $\|T_{n-i,n}(t)\| \leq \|T(t)\| \leq M e^{wt}$, we obtain that $v_k(\cdot) \rightarrow 0$ uniformly on compacts of $\mathbb{R}^+$.

□

Example 1

Let A be the generator of a strongly continuous semigroup $(T(t))$ on a Banach space. Consider the well-posed Cauchy problem

$$\begin{aligned} x^{(3)}(t) - Ax^{(2)}(t) + cx^{(1)}(t) - cAx(t) &= 0 \\ x(0) = x_0, \quad x^{(1)}(0) = x_1, \quad x^{(2)}(0) = x_2, \quad x_i &\in D(A) \end{aligned}$$

where $0 \neq c$ is any complex number.

In this case

$$R_{\lambda} = (\lambda^2(\lambda - A) + c\lambda I - cA)^{-1} = (\lambda^2 + c)^{-1} R(\lambda, A).$$

Define $z_k := - \sum_{j=0}^{k-1} B_{k-1-j} x_j + x_k.$

Then $z_0 = x_0$, $z_1 = -Ax_0 + x_1$, $z_2 = cx_0 - Ax_1 + x_2$
and the unique solution $v(\cdot)$ is given by

$$\begin{aligned} & (2\pi i)^{-1} \int_{\gamma} e^{\lambda t} (\lambda^2 + c)^{-1} (\lambda^2 R_{\lambda} z_0 + \lambda R_{\lambda} z_1 + R_{\lambda} z_2) d\lambda \\ &= (2\pi i)^{-1} \int_{\gamma} e^{\lambda t} (\lambda^2 + c)^{-1} (\lambda x_0 + x_1 + R(\lambda, A)(cx_0 + x_2)) d\lambda. \end{aligned}$$

By

$$a) \quad (\lambda^2 + c)^{-1} = \sum_{n=0}^{\infty} (-c\lambda^{-2})^n,$$

$$b) \quad (2\pi i)^{-1} \int_{\gamma} \lambda^{-(k+1)} e^{\lambda t} d\lambda = \frac{1}{k!} t^k,$$

$$\begin{aligned} c) \quad & (2\pi i)^{-1} \int_{\gamma} \lambda^{-(n+1)} e^{\lambda t} R(\lambda, A)x d\lambda \\ &= \int_0^t \frac{1}{n!} (t-s)^n T(s)x ds \quad \text{and} \end{aligned}$$

$$d) \quad \sum_{n=0}^{\infty} (-c)^n \frac{1}{2n!} t^{2n} = \cosh \sqrt{-c} t$$

$$\sum_{n=0}^{\infty} (-c)^n \frac{1}{2n+1!} t^{2n+1} = \frac{1}{\sqrt{-c}} \sinh \sqrt{-c} t$$

we obtain that the solution $v(\cdot)$ can be written as:

$$(\cosh \sqrt{-c} t)x_0 + \frac{1}{\sqrt{-c}} (\sinh \sqrt{-c} t)x_1 +$$

$$\frac{1}{\sqrt{-c}} \int_0^t (\sinh \sqrt{-c} (t-s)) T(s)(cx_0 + x_2) ds.$$

Example 2

Let A be the generator of a strongly continuous semigroup $(T(t))$ on a Banach space. Consider the well-posed Cauchy problem

$$\begin{aligned} x^{(3)}(t) - Ax^{(2)}(t) - aAx^{(1)}(t) \\ - \frac{3}{4}a^2x^{(1)}(t) - \frac{1}{4}a^2Ax(t) - \frac{1}{4}a^3x(t) = 0 \end{aligned}$$

$$x(0) = x_0, \quad x^{(1)}(0) = x_1, \quad x^{(2)}(0) = x_2, \quad x_0, x_1, x_2 \in D(A)$$

where $0 \neq a$ is any complex number.

$$\text{Then} \quad R_\lambda = (\lambda + \frac{a}{2})^{-2} R(\lambda - a, A), \quad z_0 = x_0,$$

$$z_1 = -Ax_0 + x_1, \quad z_2 = -A(ax_0 + x_1) + x_2 - \frac{3}{4}a^2x_0$$

and the solution is given by

$$\frac{1}{(2\pi i)} \int_\gamma e^{\lambda t} (\lambda + \frac{a}{2})^{-2} [\lambda^2 R(\lambda - a, A) z_0 + \lambda R(\lambda - a, A) z_1 + R(\lambda - a, A) z_2] d\lambda.$$

With similar calculations as in the previous example we get as the unique solution of the Cauchy problem:

$$\begin{aligned} e^{-a/2 \cdot t} \left[\int_0^t (t-s) e^{3a/2 \cdot s} T(s) \left(\frac{1}{4}a^2x_0 + ax_1 + x_2 \right) ds \right. \\ \left. + x_0 + t \left(\frac{a}{2}x_0 + x_1 \right) \right]. \end{aligned}$$

6. DAMPED SECOND ORDER CAUCHY PROBLEMS

In this section we investigate damped second order Cauchy problems

$$\begin{aligned} u^{(2)}(t) - Au^{(1)}(t) - Bu(t) &= f(t, u(t)) \\ u(0) &= x, \quad u^{(1)}(0) = y \end{aligned}$$

which we will call semilinear if $f(\cdot, \cdot) \neq 0$ and inhomogeneous if $f(\cdot, u(\cdot))$ is independent from $u(\cdot)$, i.e. $f(t, u(t)) = g(t)$ for some function $g(\cdot) : \mathbb{R}^+ \rightarrow E$.

We show how the problem above is related to the initial value problem

$$u^{(1)}(t) = Au(t) + \int_0^t (Bu(s) + f(s, u(s))) ds, \quad u(0) = x$$

which will be called the integrated Cauchy problem.

This section is divided into two main parts.

In part one we investigate the problems above for the case that A is an unbounded closed operator and B is a closed operator with $D(A) \subset D(B)$.

In part two we investigate cases in which A is bounded, B is densely defined with non empty resolvent set and $AR(\lambda, B) = R(\lambda, B)A$ on $D(A)$.

Unbounded A , closed B , $D(A) \subset D(B)$

We start with a version of lemma 10 for $n = 1$.

Lemma 13 Let A be the generator of a strongly continuous semigroup on E and let B be a closed linear operator on E with $D(A) \subset D(B)$.

Then there are families $(T_i(t))$, $1 \leq i \leq 4$, $t \geq 0$, of bounded linear operators on E such that the semigroup $r(t)$ generated by $A := \begin{bmatrix} A & I \\ B & 0 \end{bmatrix}$ is given by

$$\begin{bmatrix} T_1(t) & T_2(t) \\ T_3(t) & T_4(t) \end{bmatrix} .$$

For every $1 \leq i \leq 4$ and every $x \in E$ the map $t \rightarrow T_i(t)x$ is continuous and

(i) $T_1(0) = I$, $T_1(t) : D(A) \rightarrow D(A)$, $t \rightarrow T_1(t)x$ is continuously differentiable for every $x \in D(A)$ and $T_1'(t)x = AT_1(t)x + T_3(t)x = T_1(t)Ax + T_2(t)Bx$.

(ii) $T_2(0) = 0$, $T_2(t) : E \rightarrow D(A)$, $t \rightarrow T_2(t)x$ is continuously differentiable for every $x \in E$ and $T_2'(t)x = T_1(t)x = AT_2(t)x + T_4(t)x$.

(iii) $T_3(0) = 0$, $t \rightarrow T_3(t)x$ is continuously differentiable for every $x \in D(A)$ and $T_3'(t)x = BT_1(t)x = T_3(t)Ax + T_4(t)Bx$.

(iv) $T_4(0) = I$, $t \rightarrow T_4(t)x$ is continuously differentiable for every $x \in E$ and
 $T_4'(t)x = BT_2(t)x = T_3(t)x$.

(v) $T_1(t+s) = T_1(s)T_1(t) + T_2(s)T_3(t)$
 $T_2(t+s) = T_1(s)T_2(t) + T_2(s)T_4(t)$
 $T_3(t+s) = T_3(s)T_1(t) + T_4(s)T_3(t)$
 $T_4(t+s) = T_3(s)T_2(t) + T_4(s)T_4(t)$.

For a proof of statements (i) - (iv) see lemma 10.
 Statement (v) follows from $\Gamma(t+s) = \Gamma(t)\Gamma(s)$.

□

Corollary 14

Let A be the generator of a strongly continuous semigroup on E and let B be a closed linear operator on E with $D(A) \subset D(B)$. Then there is a $w \in \mathbb{R}$ such that for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > w$

$$R_\lambda := (\lambda(\lambda - A) - B)^{-1}$$

exists. R_λ is the Laplace transform of a strongly differentiable family $W(t)$ of bounded linear operators on E , i.e.

$$R_\lambda x = \int_0^\infty e^{-\lambda t} W(t)x \, dt$$

for every $x \in E$ and every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > w$. $W(t)$ is given as the inverse Laplace transform of R_λ , i.e.

$$W(t)x = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} R_{\lambda} x \, d\lambda$$

for every $x \in E$. The family $(W(t))$ has the following properties:

- (i) $W(0) = 0$,
- (ii) $W(\cdot) : E \rightarrow D(A)$, $BW(\cdot)x$ is continuous for every $x \in E$, $W'(\cdot) : D(A) \rightarrow D(A)$ and $W'(\cdot)$ is continuously differentiable for every $x \in D(A)$,
- (iii) $W'(t)x = AW(t)x + \int_0^t BW(s)x \, ds + x$,
- (iv) $W^{(2)}(t)x = AW'(t)x + BW(t)x$.

The semigroup generated by $\Delta := \begin{bmatrix} A & J \\ B & 0 \end{bmatrix}$ is given by

$$\begin{bmatrix} W'(t) & W(t) \\ BW(t) & \int_0^t BW(s) \, ds + I \end{bmatrix}.$$

Proof:

As we have seen in the proof of proposition 8 the resolvent of Δ is given by

$$\begin{bmatrix} \lambda R_{\lambda} & R_{\lambda} \\ BR_{\lambda} & (\lambda - A) R_{\lambda} \end{bmatrix}.$$

Define $W(t) := T_2(t)$ (see lemma 13). Then the statements of the corollary follow from lemma 13 and the

fact that the Laplace transform of a strongly differentiable operator family is invertible for every $x \in E$.

□

Corollary 15

Let A be the generator of a strongly continuous semigroup on E and let B be a closed linear operator on E with $D(A) \subset D(B)$. Then, for every $x \in D(A)$ and $y \in E$, the integrated Cauchy problem

$$\begin{aligned} u^{(1)}(t) &= Au(t) + \int_0^t Bu(s) ds + y \\ u(0) &= x \end{aligned}$$

has a unique solution $u(\cdot)$ in $C^1(\mathbb{R}^+, E)$. The solution is given by

$$W'(t)x + W(t)y = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} R_{\lambda}(\lambda x + y) d\lambda$$

where R_{λ} , $W(t)$ are defined as in corollary 11.

Proof:

Existence: Define

$$\begin{aligned} u(t) &:= W'(t)x + W(t)y = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} R_{\lambda}(\lambda x + y) d\lambda \\ &= T_1(t)x + T_2(t)y \end{aligned}$$

$(T_1(t), T_2(t))$ as in lemma 13). Then $u(0) = x$, $Bu(\cdot)$ is continuous and

$$\begin{aligned} u^{(1)}(t) &= AT_1(t)x + T_3(t)x + AT_2(t)y + T_4(t)y \\ &= Au(t) + \int_0^t Bu(s)ds + y. \end{aligned}$$

Uniqueness: Let $v(\cdot)$ be a solution of the integrated problem with $x = y = 0$, i.e. $v(\cdot) : \mathbb{R}^+ \rightarrow D(A)$ is in $C^1(\mathbb{R}^+, E)$, $Bv(\cdot)$ is integrable and

$$v^{(1)}(t) = Av(t) + \int_0^t Bv(s)ds, \quad v(0) = 0.$$

Define $w(t) := \int_0^t v(s) ds$. Then

- a) $w(t) \in D(B)$ for every $t \geq 0$ and $w(0) = 0$,
- b) $w'(t) \in D(A)$ for every $t \geq 0$ and $w'(0) = 0$,
- c) $Bw(\cdot)$ is continuous and

$$w^{(2)}(t) - Aw^{(1)}(t) - Bw(t) = 0.$$

Therefore, by theorem 7, $w(\cdot) = 0$ or, by the continuity of $v(\cdot)$, $v(t) = 0$ for every $t \geq 0$. Hence every solution of the integrated problem is unique.

□

Theorem 16

Let A be the generator of a strongly continuous semigroup on E and let B be a closed linear operator on E with $D(A) \subset D(B)$. Let f be a continuous function from $\mathbb{R}^+$ into E and let

$$\begin{bmatrix} T_1(t) & T_2(t) \\ T_3(t) & T_4(t) \end{bmatrix}$$

be the semigroup generated by $\Delta := \begin{bmatrix} A & I \\ B & 0 \end{bmatrix}$.

Then the following statements are equivalent:

(i) $t \rightarrow \int_0^t T_1(t-s)f(s)ds$ is continuously differentiable.

(ii) the inhomogeneous damped second order Cauchy problem

$$u^{(2)}(t) - Au^{(1)}(t) - Bu(t) = f(t)$$

$$u(0) = x, \quad u^{(1)}(0) = y$$

has for every $x, y \in D(A)$ a solution $u(\cdot)$ in $C^2(\mathbb{R}^+, E)$ such that $Bu(\cdot)$ is continuous.

If $BR_\lambda = R_\lambda B$ on $D(A)$, then every solution is unique. Otherwise the solution is unique among all solutions $u(\cdot)$ for which $Bu(\cdot)$ is a continuous function.

Proof: (i) $\rightarrow$ (ii). Uniqueness:

If there are solutions $u(\cdot)$ and $v(\cdot)$, then

$h(\cdot) := u(\cdot) - v(\cdot)$ is a solution of

$$u^{(2)}(t) - Au^{(1)}(t) - Bu(t) = 0, \quad u(0) = u^{(1)}(0) = 0.$$

If $BR_\lambda = R_\lambda B$ on $D(A)$ or if $Bh(\cdot)$ is continuous, then,

by theorem 7 and 12, $h(\cdot) = 0$. But then $u(\cdot) = v(\cdot)$.

Existence: Define

$$x(t) := T_1(t)x + T_2(t)(y - Ax) + \int_0^t T_2(t-s)f(s) \, ds,$$

$$w(t) := \int_0^t T_2(t-s)f(s) \, ds.$$

By lemma 13 we obtain that $Bx(\cdot)$ is a continuous function and

$$\begin{aligned} \frac{1}{h} (w(t+h) - w(t)) &= \frac{1}{h} (T_1(h) - I)w(t) \\ &+ \frac{1}{h} T_2(h) \int_0^t T_4(t-s)f(s) \, ds \\ &+ \frac{1}{h} \int_t^{t+h} T_2(t+h-s)f(s) \, ds. \end{aligned}$$

From $AT_2(\cdot) = T_4(\cdot) - T_1(\cdot)$ we see that $AT_2(\cdot)$ is continuous and by the closedness of A we get

$w(\cdot) : \mathbb{R}^+ \rightarrow D(A)$. Therefore the right hand side of the equation above converges as $h \rightarrow 0$, $x(\cdot)$ is continuously differentiable and

$$x^{(1)}(t) = Ax(t) + T_3(t)x + T_4(t)(y - Ax) + \int_0^t T_4(t-s)f(s) \, ds$$

Now $x^{(1)}(\cdot)$ is continuously differentiable if and only if $Ax(\cdot)$ is. But

$$Ax(t) = T_2(t)Bx + T_1(t)y - T_3(t)x - T_4(t)(y-Ax)$$

$$- \int_0^t T_4(t-s)f(s) \, ds + \int_0^t T_1(t-s)f(s) \, ds$$

and therefore $Ax(\cdot)$ is continuously differentiable if and only if

$$t \mapsto \int_0^t T_1(t-s)f(s) \, ds$$

is continuously differentiable. By (i) and the closedness of A we obtain $\frac{d}{dt} Ax(t) = x^{(1)}(t)$ and

$$\begin{aligned} x^{(2)}(t) &= Ax^{(1)}(t) + T_3'(t)x + T_4'(t)(y-Ax) \\ &\quad + \frac{d}{dt} \int_0^t T_4(t-s)f(s) \, ds \end{aligned}$$

$$\begin{aligned} &= Ax^{(1)}(t) + BT_1(t)x + BT_2(t)(y-Ax) \\ &\quad + B \int_0^t T_2(t-s)f(s) \, ds + f(t) \end{aligned}$$

$$= Ax^{(1)}(t) + Bx(t) + f(t)$$

(ii) $\Rightarrow$ (i). Let $x(\cdot)$ be a solution of the damped, inhomogeneous problem such that $Bx(\cdot)$ is continuous. Then $Ax(\cdot)$ is continuously differentiable. Define

$u(\cdot), F(\cdot) : \mathbb{R}^+ \rightarrow E \times E$ by

$$u(t) := (x(t), x^{(1)}(t) - Ax(t))$$

$$F(t) := (0, f(t))$$

The function $u(\cdot)$ is continuously differentiable and

$$u^{(1)}(t) = Au(t) + F(t) \quad , \quad u(0) = (x, y - Ax).$$

By [39], p.106, we obtain

$$u(t) = T(t)(x, y - Ax) + \int_0^t T(t-s)F(s) \, ds \quad \text{or}$$

$$x(t) = T_1(t)x + T_2(t)(y - Ax) + \int_0^t T_2(t-s)f(s) \, ds.$$

But $x(\cdot)$ is twice continuously differentiable and, as seen in the first part of the proof, this is the case if and only if $t \mapsto \int_0^t T_1(t-s)f(s) \, ds$ is continuously differentiable.

□

Lemma 17

Let $\{V(t), t \geq 0\}$ be a strongly continuous family of bounded operators on E and let $f(\cdot)$ be an element of $C^1(\mathbb{R}^+, E)$. Then

$$w(t) := \int_0^t V(t-s)f(s) \, ds$$

is continuously differentiable and

$$w^{(1)}(t) = \int_0^t V(t-s)f'(s) \, ds + V(t)f(0).$$

Proof:

$$\begin{aligned} w(t) &= \int_0^t V(t-s) \left[\int_0^s f'(r) dr + f(0) \right] ds \\ &= \int_0^t \int_0^s V(t-s) f'(r) dr ds + \int_0^t V(s) f(0) ds . \end{aligned}$$

Substituting $s = -y+t, r = x-y$

or $x = r+y = r+t-s, y = t-s,$

we obtain

$$w(t) = \int_0^t \int_y^t V(y) f'(x-y) dx dy + \int_0^t V(s) f(0) ds .$$

The function $V(y) f'(x-y)$ is integrated over the triangle $0 \leq y \leq x \leq t$. Changing the direction of integration we get

$$w(t) = \int_0^t \int_0^x V(y) f'(x-y) dy dx + \int_0^t V(s) f(0) ds .$$

The righthand side of the last equation is continuously differentiable and

$$w^{(1)}(t) = \int_0^t V(t-s) f'(s) ds + V(t) f(0) .$$

□

Lemma 18

Let $f(\cdot)$ be an integrable function and $\int_0^t T_1(t-s) f(s) ds \in D(A)$ for every $t \geq 0$. Then $t \rightarrow \int_0^t T_1(t-s) f(s) ds$ is continuously differentiable.

Proof: The statement follows from lemma 13 and the following considerations:

Define $w(t) := \int_0^t T_1(t-s)f(s) ds$. Then

$$\begin{aligned} \frac{1}{h} (w(t+h) - w(t)) &= \frac{1}{h} (T_1(h) - I)w(t) \\ &+ \frac{1}{h} T_2(h) \int_0^t T_3(t-s)f(s) ds \\ &+ \frac{1}{h} \int_t^{t+h} T_1(t+h-s)f(s) ds. \end{aligned}$$

Therefore, if $w(t) \in D(A)$, then $w(\cdot)$ is differentiable in t and

$$w^{(1)}(t) = Aw(t) + \int_0^t T_3(t-s)f(s) ds + f(t).$$

□

Corollary 19

Let A be the generator of a strongly continuous semigroup on E and let B be a closed linear operator on E with $D(A) \subset D(B)$. Let f be a continuous function from $\mathbb{R}^+$ into E such that either

- (i) $f(\cdot)$ is continuously differentiable or
- (ii) $f(\cdot) : \mathbb{R}^+ \rightarrow D(A)$ and $Af(\cdot), Bf(\cdot)$ are integrable.

Then, for every $x, y \in D(A)$, the inhomogeneous, damped second order Cauchy problem

$$\begin{aligned} u^{(2)}(t) - Au^{(1)}(t) - Bu(t) &= f(t) \\ u(0) &= x, \quad u^{(1)}(0) = y \end{aligned}$$

has a solution $u(\cdot)$ in $C^2(\mathbb{R}^+, E)$ given by

$$W'(t)x + W(t)(y - Ax) + \int_0^t W(t-s)f(s) ds$$

where $W(t)$ is defined as in corollary 14. If $BR_\lambda = R_\lambda B$ on $D(A)$, then every solution is unique. Otherwise the solution is unique among all solutions $u(\cdot)$ for which $Bu(\cdot)$ is a continuous function.

Proof:

(i) If $f(\cdot)$ is continuously differentiable, then, by lemma 17, $t \mapsto \int_0^t T_1(t-s)f(s)ds$ is continuously differentiable. Hence the statement of the corollary follows from theorem 16.

(ii) Let $f(\cdot)$ be a function from $\mathbb{R}^+$ into $D(A)$ such that $Af(\cdot)$ and $Bf(\cdot)$ are integrable. From the equation

$$AT_1(t-s)f(s) = T_1(t-s)Af(s) + T_2(t-s)Bf(s) - T_3(t-s)f(s)$$

we obtain $\int_0^t T_1(t-s)f(s) ds \in D(A)$ for every $t \geq 0$. By lemma 18 and theorem 16 the statement of the corollary follows.

□

Proposition 20

Let A be the generator of a strongly continuous semigroup on E and let B be a closed linear operator on E with $D(A) \subset D(B)$. Let f be a continuous function from $\mathbb{R}^+$ into E . Then, for every $x \in D(A)$, the integrated Cauchy problem

$$u^{(1)}(t) = Au(t) + \int_0^t (Bu(s) + f(s)) ds, \quad u(0) = x$$

has a unique solution $u(\cdot)$ in $C^1(\mathbb{R}^+, E)$. The solution is given by

$$W'(t)x + \int_0^t W(t-s)f(s) ds$$

where $W(t)$ is defined as in corollary 14.

Proof:

$$\begin{aligned} \text{Define } x(t) &:= W'(t)x + \int_0^t W(t-s)f(s) ds \\ &= T_1(t)x + \int_0^t T_2(t-s)f(s) ds. \end{aligned}$$

Then, as seen in the proof of theorem 16, $x(\cdot)$ is continuously differentiable, $x(t) \in D(A)$ for every $t \geq 0$ and

$$\begin{aligned} x^{(1)}(t) &= Ax(t) + T_3(t)x + \int_0^t T_4(t-s)f(s) ds \\ &= Ax(t) + \int_0^t (BT_1(s)x + T_4(t-s)f(s)) ds. \end{aligned}$$

We have to show that

$$\int_0^t T_4(t-s)f(s) ds = \int_0^t (B \int_0^s T_2(s-r)f(r) dr + f(s)) ds.$$

Therefore define $g(t) := \int_0^t T_4(t-s)f(s) \, ds$ and

$$h(t) := \int_0^t (B \int_0^s T_2(s-r)f(r) \, dr + f(s)) \, ds .$$

Then $g(0) = h(0)$ and

$$h^{(1)}(t) = B \int_0^t T_2(t-r)f(r) \, dr + f(t) .$$

Now, by lemma 13,

$$\begin{aligned} \frac{1}{h}(g(t+h) - g(t)) &= \frac{1}{h}(T_4(h) - I)g(t) \\ &+ \frac{1}{h}T_3(h) \int_0^t T_2(t-s)f(s) \, ds \\ &+ \frac{1}{h} \int_t^{t+h} T_4(t+h-s)f(s) \, ds . \end{aligned}$$

Therefore $g(\cdot)$ is continuously differentiable and

$$g^{(1)}(t) = B \int_0^t T_2(t-s)f(s) \, ds + f(t) = h^{(1)}(t) .$$

This proves that $x(\cdot)$ is a solution of the integrated problem. Furthermore

$$\begin{aligned} Bx(t) &= BT_1(t)x + \int_0^t BT_2(t-s)f(s) \, ds \\ &= T_3'(t)x + \int_0^t T_3(t-s)f(s) \, ds . \end{aligned}$$

Hence $Bx(\cdot)$ is even continuous and not only integrable. Suppose there is another solution $v(\cdot)$ of the integrated problem. Then $h(\cdot) = x(\cdot) - v(\cdot)$ is a solution of

$$u^{(1)}(t) = Au(t) + \int_0^t Bu(s)ds, \quad u(0) = 0.$$

By corollary 15 it follows that $h(t) = 0$ for every $t \geq 0$.

□

In the same way as in theorem 16 we characterize existence of solutions of a semilinear damped second order equation.

Proposition 21

Let A be the generator of a strongly continuous semigroup on E and let B be a closed linear operator on E with $D(A) \subset D(B)$. Let f be a function from $[0, T] \times E$ into E such that $f(\cdot, u(\cdot))$ is a continuous function from $[0, T]$ into E whenever $u(\cdot) \in C([0, T], E)$. Define $W(t)$ ($= T_2(t)$) as in corollary 14 (as in lemma 13). The following statements are equivalent:

- (i) For every $x, y \in D(A)$ the semilinear damped Cauchy problem

$$\begin{aligned} u^{(2)}(t) - Au^{(1)}(t) - Bu(t) &= f(t, u(t)) \\ u(0) &= x, \quad u^{(1)}(0) = y \end{aligned}$$

has a solution $u(\cdot)$ in

$$C^2([0, T], E) \cap C^1([0, T], [D(A)]) .$$

(ii) For every $x, y \in D(A)$ the integral equation

$$u(t) = W'(t)x + W(t)(y - Ax) + \int_0^t W(t-s)f(s, u(s)) \, ds$$

has a solution $u(\cdot)$ in $C([0, T], E)$ such that

$$t \mapsto \int_0^t W'(t-s)f(s, u(s)) \, ds$$

is continuously differentiable on $[0, T]$.

Proof:

(i) $\rightarrow$ (ii). Let $u(\cdot)$ be in $C^2([0, T], E) \cap C^1([0, T], [D(A)])$. Then $u^{(1)}(\cdot)$ maps $[0, T]$ into $D(A)$ and $Au(\cdot)$ is continuously differentiable. Let $\Delta := \begin{bmatrix} A & I \\ B & 0 \end{bmatrix}$. Define

$$x(t) = (x_1'(t), x_2(t)) := (u(t), u^{(1)}(t) - Au(t))$$

and $F: [0, T] \times (E \times E) \rightarrow E \times E$, $F(t, x(t)) := (0, f(t, x_1(t)))$.

Then $x(\cdot): [0, T] \rightarrow D(\Delta)$, $x(\cdot) \in C^1([0, T], E \times E)$ and

$$x^{(1)}(t) = \Delta x(t) + F(t, x(t)), \quad x(0) = (x, y - Ax).$$

Therefore $x(\cdot)$ is a continuously differentiable solution of the integral equation

$$u(t) = \Gamma(t)u(0) + \int_0^t \Gamma(t-s)F(s, u(s)) \, ds$$

where $\Gamma(t)$ is the semigroup generated by Δ (see lemma 13). This implies that $x_1(\cdot) \in C^1([0,T],E)$ is a solution of

$$\begin{aligned} u(t) &= T_1(t)x + T_2(t)(y-Ax) + \int_0^t T_2(t-s)f(s,u(s)) \, ds \\ &= W'(t)x + W(t)(y-Ax) + \int_0^t W(t-s)f(s,u(s)) \, ds . \end{aligned}$$

By definition of $x_1(\cdot)$ we have $x_1(\cdot) \in C^2([0,T],E)$. In the same way as in the proof of theorem 16 we obtain

$$\begin{aligned} x_1^{(1)}(t) &= T_2(t)Bx + T_1(t)y + \int_0^t T_1(t-s)f(s,x_1(s)) \, ds \\ &= W(t)Bx + W'(t)y + \int_0^t W'(t-s)f(s,x_1(s)) \, ds . \end{aligned}$$

From the continuous differentiability of $x_1^{(1)}(\cdot)$, $W(\cdot)Bx$ and $W'(\cdot)y$ we conclude that

$$t \mapsto \int_0^t W'(t-s)f(s,u(s)) \, ds$$

is continuously differentiable on $[0,T]$.

The proof of (ii) $\rightarrow$ (i) is the same as the corresponding part of the proof of theorem 16 and is omitted.

□

Corollary 22

Let A be the generator of a strongly continuous semigroup on E and let B be a closed linear operator on E with $D(A) \subset D(B)$. Let f be a function from $[0, T] \times E$ into E such that $f(\cdot, u(\cdot))$ is a continuous function from $[0, T]$ into E whenever $u(\cdot) \in C([0, T], E)$. Define $W(t)$ ($= T_2(t)$) as in corollary 14 (as in lemma 13) and let $x \in D(A)$. The following statements are equivalent:

(i) The integral equation

$$u(t) = W'(t)x + \int_0^t W(t-s)f(s, u(s)) \, ds$$

has a solution $v(\cdot)$ in $C([0, T], E)$.

(ii) The integrated Cauchy problem

$$u^{(1)}(t) = Au(t) + \int_0^t (Bu(s) + f(s, u(s))) \, ds$$

$$u(0) = x$$

has a solution $v(\cdot)$ with the properties
 $v(\cdot): [0, T] \rightarrow D(A)$, $v(\cdot) \in C^1([0, T], E)$ and
 $Bv(\cdot) \in C([0, T], E)$.

Proof:

(i) $\rightarrow$ (ii). Let $v(\cdot)$ be a solution of the integral equation, i.e.

$$v(t) = T_1(t)x + \int_0^t T_2(t-s)f(s, v(s)) \, ds.$$

Then $v(\cdot): [0, T] \rightarrow D(A)$, $v(\cdot) \in C^1([0, T], E)$ and

$$\begin{aligned} v^{(1)}(t) &= Av(t) + T_3(t)x + \int_0^t T_4(t-s)f(s, v(s)) \, ds \\ &= Av(t) + \int_0^t BT_1(s)x + T_4(t-s)f(s, v(s)) \, ds . \end{aligned}$$

In the same way as in the proof of proposition 20 we show

$$T_4(t-s)f(s, v(s)) = B \int_0^s T_2(s-r)f(r, v(r)) \, dr + f(s, v(s)) .$$

Therefore $v(\cdot)$ is a solution of the integrated Cauchy problem and

$$Bv(\cdot) = T_3'(t)x + \int_0^t T_3(t-s)f(s, v(s)) \, ds$$

is a continuous function.

(ii) $\rightarrow$ (i). Let $v(\cdot)$ be a solution of the integrated Cauchy problem with the properties stated in (ii). Then the function

$$t \rightarrow x(t) := (v(t), \int_0^t Bv(s) + f(s, v(s)) \, ds)$$

is in $C^1([0, T], ExE)$, $x(\cdot): [0, T] \rightarrow D(\Delta)$ ($\Delta = \begin{bmatrix} A & I \\ B & 0 \end{bmatrix}$)
and

$$\Delta x(t) = (v^{(1)}(t), Bv(t)) = x^{(1)}(t) - F(t, x(t))$$

where $F(t, x(t)) := (0, f(t, x_1(t)))$

and

$$x(t) := (x_1(t), x_2(t)) .$$

Therefore $x(\cdot)$ is a solution of

$$x^{(1)}(t) = \Delta x(t) + F(t, x(t)) , \quad x(0) = (x, 0)$$

and hence a solution of

$$x(t) = \Gamma(t)(x, 0) + \int_0^t \Gamma(t-s)F(s, x(s)) \, ds ,$$

where $\Gamma(t)$ is the semigroup generated by Δ . This implies

$$x_1(t) = T_1(t)x + \int_0^t T_2(t-s)f(s, x_1(s)) \, ds .$$

But $x_1(t) = v(t)$ and therefore $v(\cdot)$ is a solution of the integral equation.

□

Example 3

Let A be the generator of a strongly continuous semigroup $(T(t))$ on a Banach space. In many applications (e.g. the linear strongly damped wave equation, the Klein-Gordon equation, the vibrating beam equation etc., see page 53, [4] and [44]) the damped second order Cauchy problem appears in the form

$$\begin{aligned} (\#) \quad u^{(2)}(t) - (aA + bI)u^{(1)}(t) - (cA + dI)u(t) &= f(t) \\ u(0) &= x, \quad u^{(1)}(0) = y \end{aligned}$$

where a is a positive real number, b, c, d are arbitrary complex numbers and $f(\cdot)$ is a function satisfying, for example, the conditions of corollary 19. As proved in theorem 12, the homogeneous Cauchy problem (#) is well-posed. By corollary 19, for every $x, y \in D(A)$ the unique solution of (#) is given by

$$W'(t)x + W(t)(y - Ax) + \int_0^t W(t-s)f(s) ds$$

where $W(t)$ is the inverse Laplace transform of

$$\begin{aligned} \lambda \rightarrow R_\lambda &= (\lambda(\lambda - (aA + bI)) - (cA + dI))^{-1} \\ &= \frac{1}{a\lambda + c} R\left(\frac{1}{a\lambda + c}(\lambda^2 - \lambda b - d), A\right). \end{aligned}$$

In order to determine $W(t)$ we will recall two elementary operations from Laplace transform theory (see [12]):

(1) If $F(t)$ is the inverse Laplace transform of $f(\lambda)$, then $\frac{1}{a} e^{bt/a} F\left(\frac{t}{a}\right)$ is the inverse Laplace transform of $f(a\lambda - b)$ if $a > 0$ and $b \in \mathbb{C}$.

(2) If $F(t)$ is the inverse Laplace transform of $f(\lambda)$, then $\int_0^t J_0(2\sqrt{(t-s)s}) F(s) ds$ is the inverse Laplace transform of $\frac{1}{\lambda} f\left(\lambda + \frac{1}{\lambda}\right)$, where $J_0(t)$ stands for the Bessel function

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \left(\frac{t}{2}\right)^{2n}.$$

Now suppose $q := (\sqrt{-d + \frac{c^2}{a^2} + \frac{bc}{a}})^{-1} > 0$. Define $f := \frac{1}{aq}$, $g := \frac{f}{q}$, $h := -\frac{1}{a}(b + \frac{2c}{a})$ and $i := \frac{h}{f}$. Then, by the elementary operations (1) and (2) the following is true:

- $\frac{g}{a} R(\lambda, A)x$ is the Laplace transform of $\frac{g}{a} T(t)x$,
- $\frac{g}{a} R(f(\lambda+i), A)x$ is the Laplace transform of $\frac{g}{af} e^{-it} T(\frac{t}{f})x$,
- $\frac{g}{a} \cdot \frac{1}{\lambda} R(f(\lambda + \frac{1}{\lambda} + i), A)x$ is the Laplace transform of $\int_0^t J_0(2\sqrt{(t-s)s}) \frac{g}{af} e^{-is} T(\frac{s}{f})x ds$,
- $\frac{1}{a} \cdot \frac{1}{\lambda} R(f(q\lambda + \frac{1}{q\lambda} + i), A)x = \frac{1}{a} \cdot \frac{1}{\lambda} R(\frac{1}{a}\lambda + \frac{g}{\lambda} + h, A)x$ is the Laplace transform of $\frac{1}{af} \int_0^{t/q} J_0(2\sqrt{(\frac{t}{q}-s)s}) e^{-is} T(\frac{s}{f})x ds$,
- $\frac{1}{a} \cdot \frac{1}{\lambda + c/a} R(\frac{1}{a}(\lambda + \frac{c}{a}) + \frac{g}{\lambda + c/a} + h, A)x = \frac{1}{a\lambda + c} R(\frac{1}{a\lambda + c}(\lambda^2 - \lambda b - d), A)$

is the Laplace transform of

$$\begin{aligned} & \frac{1}{af} e^{-(c/a)t} \int_0^{t/q} J_0(2\sqrt{(\frac{t}{q}-s)s}) e^{-is} T(\frac{s}{f})x ds \\ &= e^{-(c/a)t} \int_0^t J_0(\frac{2}{q}\sqrt{(t-s)s}) e^{-ahs} T(as)x ds \\ &= e^{-(c/a)t} \int_0^t J_0[2\sqrt{-d + \frac{c^2}{a^2} + \frac{bc}{a}} \cdot \sqrt{(t-s)s}] e^{(b+2(c/a))s} T(as)x ds. \end{aligned}$$

Therefore, for $e > 0$ and every $x \in E$, $W(t)x$ is given by the last expression. By some tedious calculations it

can be seen that this expression is a solution of (#) (with $v(0)=0$ and $v'(0)=x$) not only for $e > 0$ but for every complex numbers b, c, d and every $a > 0$. Hence, by theorem 12, $W(t)x$ is given by

$$e^{-(c/a)t} \int_0^t J_0 \left[2\sqrt{-d + \frac{c^2 + bc}{a^2 + a}} \cdot \sqrt{(t-s)s} \right] e^{(b+2(c/a))s} T(as) x ds$$

If the semigroup $(T(t))$ generated by A is known we obtain by

$$W'(t)x + W(t)(y - Ax) + \int_0^t W(t-s)f(s) ds$$

an explicit expression of the unique solution of (#).

Before continuing, a remark to the previous example will be made: In the second part of this section we will see that the undamped second order Cauchy problem

$$u^{(2)}(t) - Au(t) = 0, \quad u(0) = x, \quad u^{(1)}(0) = y$$

has a unique solution for every initial data (x, y) in $D(A)^2$ if and only if A is the generator of a certain holomorphic semigroup (i.e. A is the generator of a so-called cosine family). We conjecture that a (analytic) semigroup generator is a cosine generator if and only if the solutions $u_a(\cdot)$ of the damped equations

$$\begin{aligned} (##) \quad & u^{(2)}(t) - aAu^{(1)}(t) - Au(t) = 0 \\ & u(0) = x, \quad u^{(1)}(0) = 0, \end{aligned}$$

which are fully known by the calculations above, converge uniformly on compacts for $a \rightarrow 0$.

Bounded A , densely defined B , $\rho(B) \neq \emptyset$, $AR(\lambda, B) = R(\lambda, B)A$

We now study the damped second order Cauchy problem

$$u^{(2)}(t) - Au^{(1)}(t) - Bu(t) = 0$$

$$u(0) = x, \quad u^{(1)}(0) = y$$

when A is a bounded and B a densely defined linear operator such that $\rho(B) \neq \emptyset$ and $AR(\lambda, B) = R(\lambda, B)A$. We will see that, for every $(x, y) \in D(A)^2$, the problem has a unique solution $u(\cdot) : \mathbb{R}^+ \rightarrow D(A)$ in $C^2(\mathbb{R}^+, E)$ if and only if B is the generator of a strongly continuous cosine family.

Therefore we start with a short summary of cosine family theory.

A family of bounded linear operators $\{C(t), t \in \mathbb{R}\}$ on a Banach space E is called a strongly continuous cosine family $(C(t))$ if:

- (i) $C(0) = I$,
- (ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for $s, t \in \mathbb{R}$,
- (iii) $t \rightarrow C(t)x$ is continuous for every $x \in E$.

The generator B of the cosine family $(C(t))$ is defined by

$$Bx := C^{(2)}(0)x = \lim_{h \rightarrow 0} \frac{2}{h^2} (C(h)x - x)$$

whenever the second derivative or, equivalently, the limit exists. The domain $D(B)$ of B is the set of all $x \in E$ such that $t \rightarrow C(t)x$ is twice continuously differentiable on $\mathbb{R}$ or, equivalently, the set of those $x \in E$ such that the limit exists.

B is a densely defined, closed operator with non empty resolvent set $\rho(B)$.

For every strongly continuous cosine family $(C(t))$ there exist constants $w \geq 0$ and $M \geq 1$ such that

$$\|C(t)\| \leq M e^{w|t|} \quad \text{for every } t \in \mathbb{R}.$$

We remark that the constant w has to be greater or equal to zero. (Suppose $\|C(t)\| \leq M$ and $\|C(t)x\| \rightarrow 0$ for $t \rightarrow +\infty$ and some $x \in E$. Then, by the cosine property: $x = 2C(t)C(t)x - C(2t)x$, we obtain $x = 0$).

For every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > w$ we have $\lambda^2 \in \rho(B)$ and $\lambda R(\lambda^2, A)x$ is the Laplace transform of $C(t)x$, i.e.

$$\lambda R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t} C(t)x \, dt.$$

For every strongly continuous cosine family $(C(t))$ the associated sine family $(S(t))$ is defined by

$$S(t)x := \int_0^t C(s)x \, ds.$$

A linear operator B is the generator of a strongly continuous cosine family if and only if

- (i) B is densely defined,
- (ii) there is a $w \geq 0$ such that $\lambda^2 \in \rho(B)$ for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > w$. For every $n \in \mathbb{N}$ and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > w$ we have

$$\|d^n/d\lambda^n \lambda R(\lambda^2, A)\| \leq Mn! (\operatorname{Re} \lambda - w)^{-(n+1)}$$

However, this characterization (due to Sova, [46]) is of little practical use in possible applications. It is almost impossible to verify the norm estimate for every $n \in \mathbb{N}$ (see [30]).

Another characterization of a cosine family generator was given by Kislyfsky [28]. He proved that a densely defined linear operator B is the generator of a strongly continuous cosine family if and only if

- (i) there is a Banach space F with $D(B) \subset F \subset E$ such that the Banach space topology of F is not weaker than the topology induced in F by E .
- (ii) The operator $\begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}$ with domain $D(B) \times F$ is the generator of a strongly continuous group on the Banach space $F \times E$.

In most cases the space F , given as the closure of $D(B)$ under the norm

$$\|x\|_F := \|x\| + \sup \left\{ \left\| \frac{1}{n!} (\lambda - w)^{n+1} (-d^n/d\lambda^n) \lambda R(\lambda^2, A) x \right\|, \right. \\ \left. \lambda > w, n \in \mathbb{N} \right\},$$

(see [48]), is neither $[D(B)]$ nor E as we will see in the following proposition.

Proposition 23 Let B be an unbounded cosine family generator. Then $[D(B)] \neq F \neq E$.

Proof:

Suppose $F = E$. Then, by the Phillips perturbation theorem, $\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}$ would be a semigroup generator on $E \times E$. Hence a right half plane would be contained in the resolvent set of $\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}$. Let $\begin{bmatrix} U & V \\ W & X \end{bmatrix}$ denote the resolvent of $\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}$ in some $\lambda \neq 0$. Then it can be easily seen that W is given by $\lambda^{-2} B$. Hence B is bounded.

Suppose $F = [D(B)]$. B is a bounded operator from F into E . Then, by the Phillips perturbation theorem, $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ would be a semigroup generator on $E \times E$. Hence a right half plane would be contained in the resolvent set of $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$. Let $\begin{bmatrix} U & V \\ W & X \end{bmatrix}$ denote the resolvent of $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ in $\lambda \neq 0$. Then V is given by $\lambda^{-2} I_{E,F}$. But V is a bounded operator from E into F and therefore $E = F$. Hence B is bounded.

□

We now recall two well known relations between cosine and semigroup generators.

- (1) Every cosine generator is the generator of a holomorphic semigroup $(T(t))$ on E . $T(t)x$ is given by

$$\frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/4t} C(s)x \, ds$$

(see [14] or [19]).

(2) If B is the generator of a strongly continuous group $(T(t))$ on E , then B^2 is the generator of a strongly continuous cosine family $(C(t))$. $C(t)x$ is given by $\frac{1}{2} (T(t)x + T(-t)x)$.

For further information about cosine families and references to the literature we refer to [13],[21],[30],[47].

If B is the generator of a strongly continuous cosine family $(C(t))$, then, for every $x \in D(A)$ and $y \in F$,

$$t \mapsto C(t)x + S(t)y$$

is the unique solution of the (undamped) second order abstract Cauchy problem

$$u^{(2)}(t) - Bu(t) = 0, \quad u(0) = x, \quad u^{(1)}(0) = y.$$

Fattorini [14], Sova [46] and Da Prato - Giusti [9] proved that well-posedness of the undamped second order Cauchy problem is equivalent to B being the generator of a strongly continuous cosine family. In the next theorem a unified treatment and some generalisations of their results are given. In the proof we make use of a perturbation theorem (see [35]) which says that if B is a cosine family generator and C is bounded, then $B + C$ is a cosine family generator too.

Theorem 21

Let B be a densely defined linear operator on a Banach space E . Consider the following second order abstract Cauchy problems:

$$\begin{aligned} (\#) \quad & u^{(2)}(t) - Au^{(1)}(t) - Bu(t) = 0 \\ & u(0) = x, \quad u^{(1)}(0) = y, \end{aligned}$$

$$\begin{aligned} (\#\#) \quad & u^{(2)}(t) - Bu(t) = 0, \\ & u(0) = x, \quad u^{(1)}(0) = 0. \end{aligned}$$

The following statements are equivalent:

- (i) B is the generator of a strongly continuous cosine family.
- (ii) $\rho(B) \neq \emptyset$ and, for every $x \in D(B)$, $y \in F$ (see page 87) and every bounded operator A with $AR(\lambda, B) = R(\lambda, B)A$, $(\#)$ has a unique solution $u(\cdot): \mathbb{R} \rightarrow D(B)$, in $C^2(\mathbb{R}, E)$. For every sequence of solutions of $(\#)$ with

$$\lim_{n \rightarrow \infty} u_n(0) = \lim_{n \rightarrow \infty} u_n^{(1)}(0) = 0$$

we have $u_n(\cdot) \rightarrow 0$ uniformly on compacts of $\mathbb{R}$.

- (iii) $\rho(B) \neq \emptyset$ and, for every $x \in D(B)$, $(\#\#)$ has a unique solution $u(\cdot): \mathbb{R}^+ \rightarrow D(B)$ in $C^2(\mathbb{R}^+, E)$.

- (iv) B is closed and there exists a dense subset D such that, for every $x \in D$, (##) has a solution $u(\cdot): \mathbb{R}^+ \rightarrow D(B)$ in $C^2((0, \infty), E) \cap C^1(\mathbb{R}^+, E)$. For every sequence of solutions of (##) with $\lim_{n \rightarrow \infty} u_n(0) = 0$ we have $u_n(\cdot) \rightarrow 0$ uniformly on compacts of $\mathbb{R}^+$.
- (v) B is closed and there exists a dense subset D such that, for every $x \in D$, (##) has a solution $u(\cdot): \mathbb{R}^+ \rightarrow D(B)$ in $C^2((0, \infty), E) \cap C^1(\mathbb{R}^+, E)$. There exists a locally bounded function $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\|u(t)\| \leq f(t) \|u(0)\|$ for every solution $u(\cdot)$ of (##) in $C^2((0, \infty), E) \cap C^1(\mathbb{R}^+, E)$ and every $t \in \mathbb{R}^+$.

Proof:

(i) $\rightarrow$ (ii). Let B be a cosine family generator and A a bounded operator with $AR(\lambda, B) = R(\lambda, B)A$. Then $B + \frac{1}{4}A^2$ is a cosine family generator, $A: D(B) \rightarrow D(B)$ and the second order Cauchy problem

$$\begin{aligned} u^{(2)}(t) - (B + \frac{1}{4}A^2)u(t) &= 0 \\ u(0) &= x, \quad u^{(1)}(0) = y - \frac{1}{2}Ax \end{aligned}$$

has a unique solution $u(\cdot): \mathbb{R} \rightarrow D(B)$ in $C^2(\mathbb{R}, E)$ for every $x \in D(B)$, $y \in F$.

Define $v(t) := T(t)u(t)$ where $(T(t))$ is the group generated by $\frac{1}{2}A$. From $AR(\lambda, B) = R(\lambda, B)A$ we conclude that $T(t)R(\lambda, B) = R(\lambda, B)T(t)$, $T(t): D(B) \rightarrow D(B)$ and $BT(t) = T(t)B$ on $D(B)$.

Moreover
$$v^{(1)}(t) = \frac{1}{2}AT(t)u(t) + T(t)u^{(1)}(t)$$

$$v^{(2)}(t) = \frac{1}{2}A^2v(t) + AT(t)u^{(1)}(t) + Bv(t)$$

or
$$v^{(2)}(t) - Av^{(1)}(t) - Bv(t) = 0$$
$$v(0) = x, v^{(1)}(0) = y.$$

Suppose there is another solution $g(\cdot)$ of $(\#)$ with $g(\cdot): \mathbb{R} \rightarrow D(B)$ and $g(\cdot) \in C^2(\mathbb{R}, E)$. Then $h(\cdot) := g(\cdot) - v(\cdot)$ is a solution of

$$u^{(2)}(t) - Au^{(1)}(t) - Bu(t) = 0$$
$$u(0) = 0, u^{(1)}(0) = 0$$

and $f(\cdot) := T(\cdot)h(\cdot)$ a solution of

$$u^{(2)}(t) - (B + \frac{1}{4}A^2)u(t) = 0$$
$$u(0) = 0, u^{(1)}(0) = 0.$$

Therefore $f(\cdot) = 0$ and, by the bijectivity of $T(-t)$, $h(\cdot) = 0$ or $g(\cdot) = v(\cdot)$.

Let $h_n(\cdot)$ be a sequence of solutions of (##) such that

$$\lim_{n \rightarrow \infty} h_n(0) = \lim_{n \rightarrow \infty} h_n^{(1)}(0) = 0.$$

Define $f_n(\cdot) := T(\cdot)h_n(\cdot)$. Then the functions $f_n(\cdot)$ are solutions of

$$\begin{aligned} u^{(2)}(t) - (B + \frac{1}{4}A^2)u(t) &= 0 \\ u(0) &= h_n(0), \quad u^{(1)}(0) = h_n^{(1)}(0) - \frac{1}{2}Ah_n(0). \end{aligned}$$

But then $f_n(\cdot)$ and hence $h_n(\cdot)$ converge to zero uniformly on compacts of $\mathbb{R}$.

(ii) $\rightarrow$ (iii) is obvious.

(iii) $\rightarrow$ (iv). Let $u_n(\cdot)$ be a sequence of solutions of (##) in $C^2((0, \infty), E) \cap C^1(\mathbb{R}^+, E)$ such that

$$u_n := u_n(0) \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

Define $v_n(\cdot) := R(\lambda, B)u_n(\cdot)$. Then $v_n := F(\lambda, B)u_n \in D(B)$ and $v_n(\cdot)$ is a sequence of solutions of (##) in $C^2(\mathbb{R}^+, E)$ such that v_n and Bv_n converge to zero for $n \rightarrow \infty$.

Define $K: [D(B)] \rightarrow M_2$, $u \mapsto u(\cdot)$,

where $u(\cdot)$ is the unique solution of (##) with initial value u (for the definition of the space M_2 , see

section 1). Then K is a closed, everywhere defined linear operator. Therefore K is continuous and from the fact that v_n converges to zero in $[D(B)]$ we conclude that $v_n(\cdot) \rightarrow 0$ in M_2 .

But $u_n(\cdot) = \lambda v_n(\cdot) - v_n^{(2)}(\cdot)$. Therefore $u_n(\cdot) \rightarrow 0$ uniformly on compacts of $\mathbb{R}^+$.

(iv) $\rightarrow$ (v). Let $D' := \{ u \in E : (\#\#) \text{ has a solution } u(\cdot) \text{ in } C^2((0, \infty), E) \cap C^1(\mathbb{R}^+, E) \}$.

If $u(0) = 0$, then $u(t) = 0$ for every $t \in \mathbb{R}^+$. Therefore the solution is unique and we define for every $t \in \mathbb{R}^+$ a linear operator $R(t)$ from the (dense) subspace D' into E by

$$R(t)u := u(t),$$

where $u(\cdot)$ is the unique solution of $(\#\#)$ with initial value u .

We show that for any compact subset I of $\mathbb{R}^+$ there exists a positive real number M_I such that for every $u \in D'$ and every $t \in I$ we have

$$\|u(t)\| = \|R(t)u\| \leq M_I \|u\| = M_I \|u(0)\|.$$

This is equivalent to:

For every compact subset I of $\mathbb{R}^+$ there exists an $\varepsilon > 0$ such that for every $u \in D'$ with $\|u\| \leq \varepsilon$ and every $t \in I$ we have $\|R(t)u\| \leq 1$.

Suppose this is false. Then there is a compact subset I of $\mathbb{R}^+$ such that for every $n \in \mathbb{N}$ there is a $u_n \in D'$ with $\|u_n\| \leq \frac{1}{n}$ and a $t_n \in I$ such that

$$\|R(t_n)u_n\| = \|u_n(t_n)\| > 1.$$

This contradicts the assumption that $u_n(\cdot) \rightarrow 0$ uniformly on compacts of $\mathbb{R}^+$.

Therefore, there is a locally bounded function $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for every solution $u(\cdot)$ and every $t \in \mathbb{R}^+$:

$$\|u(t)\| \leq f(t) \|u(0)\|.$$

(v) $\rightarrow$ (i). Define D' and $R(t)$ as above. From

$$\|u(t)\| = \|R(t)u\| \leq f(t) \|u\| = f(t) \|u(0)\|$$

we see that for each $t \in \mathbb{R}^+$, $R(t)$ has a unique bounded continuation on E which we denote with the same symbol. By definition of $R(t)$, the function $t \rightarrow R(t)u$ is continuous on $\mathbb{R}^+$ for every $u \in D'$. Let $u \in E$ and $u_n \in D'$ with $u_n \rightarrow u$. Let I be a compact subset of $\mathbb{R}^+$. Then there is a constant M such that $f(t) \leq M$ for every $t \in I$. Then, for $t \in I$, we have

$$\|R(t)u - R(t)u_n\| \leq M \|u - u_n\|.$$

Since $R(\cdot)u$ is the uniform limit on compacts of the continuous functions $R(\cdot)u_n$, $R(\cdot)u$ is continuous for every $u \in E$.

For every $u \in D'$ we have $R(0)u = u$. But D' is dense in E and therefore $R(0) = I$.

Now we extend the family $R(t)$ to the whole real line by defining $R(t) := R(-t)$ for $t < 0$.

Let $u \in D'$ and define $v(t) := R(t+s)u$, $h(t) := R(t-s)u$. Then $h(\cdot) + v(\cdot)$ is a solution of $u^{(2)}(t) - Bu(t) = 0$, $u(0) = 2R(s)u$, $u^{(1)}(0) = 0$. Therefore

$$2R(t)R(s)u = h(t) + v(t) = R(t+s)u + R(t-s)u$$

for every $u \in D'$. But D' is dense and therefore $(R(t))$ is a strongly continuous cosine family on E . Let C denotes its generator.

If $u \in D'$ then $s \mapsto BR(s)u = u^{(2)}(s)$ is continuous on $(0, \infty)$. Therefore, for all t with $0 < h < t$ we have

$$\int_0^h \int_{-h}^0 R(r+s+t)u \, dr \, ds \in D(B) \text{ and}$$

$$B \int_0^h \int_{-h}^0 R(r+s+t)u \, dr \, ds = 2R(t)(R(h)u - u).$$

By the closedness of B we get

$$\int_0^h \int_{-h}^0 R(r+s)u \, dr \, ds \in D(B) \quad \text{for every } u \in E \text{ and}$$

$$B \int_0^h \int_{-h}^0 R(r+s)u \, dr \, ds = 2(R(h)u - u) .$$

Now let $u \in D(C)$. Then

$$\frac{1}{h^2} \int_0^h \int_{-h}^0 R(r+s)u \, dr \, ds \rightarrow u \quad \text{for } h \rightarrow 0 \text{ and}$$

$$B\left(\frac{1}{h^2} \int_0^h \int_{-h}^0 R(r+s)u \, dr \, ds\right) = \frac{2}{h^2}(R(h)u - u) + Cu .$$

Therefore $u \in D(B)$ and $Bu = Cu$ or $D(C) \subset D(B)$ and $B|_{D(C)} = C$.

Now we show that there is a sequence (λ_n) in $\mathbb{C}$ with $\operatorname{Re} \lambda_n \rightarrow +\infty$ and $\lambda_n \notin p\sigma(B)$. Assume this is false. Then there is an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there are $w_n \in D(B)$, $w_n \neq 0$, with $Bw_n = \lambda_n w_n$. But then, for every $n \geq n_0$, $w_n(t) := (e^{\sqrt{n}t} + e^{-\sqrt{n}t})w_n$ is a solution of (##) which contradicts $\|u(t)\| \leq M\|u(0)\|$ for every $t \in [0,1]$ and every solution $u(\cdot)$ of (##).

Assume that $D(C) \neq D(B)$. Then choose $u \in D(B)$ with $u \notin D(C)$. Let $\lambda \in \rho(C)$ and $\lambda \notin \rho\sigma(B)$. Then there is a $v \in D(C)$ with $(\lambda - C)v = (\lambda - B)u$. But $B/D(C) = C$ and therefore $(\lambda - B)(u - v) = 0$. Since $u - v \neq 0$, we obtain $\lambda \in \rho\sigma(A)$ which is a contradiction. Therefore $D(B) = D(C)$ and $B = C$.

□

7. APPENDIX

THE INVERSION OF THE LAPLACE TRANSFORM

Let $f \in C(\mathbb{R}^+, E)$ be a function of exponential order (that is, there exist constants M and w such that $\|f(t)\| \leq M e^{wt}$ for every $t \in [0, \infty)$). The Laplace transform of $f(\cdot)$, which is denoted by $R(\cdot)$ or $L(f)(\cdot)$, is given by the formula

$$R(\lambda) = L(f)(\lambda) = \int_0^\infty e^{-\lambda s} f(s) ds$$

where

$$\int_0^\infty e^{-\lambda s} f(s) ds := \lim_{t \rightarrow \infty} \int_0^t e^{-\lambda s} f(s) ds.$$

From the equation

$$\begin{aligned} \int_0^t e^{-\lambda s} f(s) ds &= e^{-(\lambda-\mu)t} \int_0^t e^{-\mu s} f(s) ds \\ &+ (\lambda-\mu) \int_0^t e^{-(\lambda-\mu)s} \int_0^s e^{-\mu r} f(r) dr ds \end{aligned}$$

it follows that if the Laplace transform of $f(\cdot)$ exists for some $\mu \in \mathbb{C}$, then it exists for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \operatorname{Re} \mu$. Therefore, for every $f \in C(\mathbb{R}^+, E)$ (of exponential order), the domain of $L(f)$ is always a (non empty) right halfplane.

The usefulness of the Laplace transform lies in the fact that it transforms certain differential equations into "algebraic" equations. For example:

Let A be a closed linear operator and suppose that the initial value problem

$$(di) \quad u'(t) = Au(t) \quad , \quad u(0) = x$$

has a solution $u(\cdot)$ in $C^1(\mathbb{R}^+, E)$ such that $u'(\cdot)$ is of exponential order. The Laplace transform of the derivative $u'(\cdot)$ is very closely related to the Laplace transform of $u(\cdot)$, more precisely: the Laplace transform of $u(\cdot)$ exists whenever the Laplace transform of $u'(\cdot)$ exists and

$$L(u')(\lambda) = \lambda L(u)(\lambda) - u(0) \quad .$$

Taking the Laplace transform of both sides of the differential equation, (di) becomes the algebraic equation

$$(ai) \quad \lambda L(u)(\lambda) - x = AL(u)(\lambda)$$

$$\text{or} \quad (\lambda - A)L(u)(\lambda) = x \quad .$$

If a right halfplane is contained in the resolvent set of A , then the Laplace transform of the solution of (di) is given as $L(u)(\lambda) = R(\lambda, A)x$. But the Laplace transform of a differentiable function is invertible (see proposi-

tion A.3 below) and we obtain by the inversion formula for the Laplace transform an expression for the solution $u(\cdot)$ of (di) in terms of the operator A , i.e.:

$$\begin{aligned} u(t) &= \frac{1}{2\pi i} \lim_{r \rightarrow \infty} \int_{p-ir}^{p+ir} e^{\lambda t} L(u)(\lambda) d\lambda \\ &= \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} R(\lambda, A) x d\lambda . \end{aligned}$$

In the first proposition we give an elementary proof of the inversion formula of the Laplace transform. The main ideas are taken from the book of G. Doetsch [12] on the classical Laplace transform.

Lemma A.1

(Lemma of Riemann-Lebesgue)

Let $f(\cdot)$ be a continuous function from an interval $[a, b]$ into a Banach space E . Then

$$\lim_{|r| \rightarrow \infty} \int_a^b e^{-irt} f(t) dt = 0 .$$

Proof:

Let $R([a, b], E)$ be the closure of the set of step functions in the Banach space of the bounded functions from $[a, b]$ into E . The family (R_r) of linear operators $R_r : R([a, b], E) \rightarrow E$, $r \in \mathbb{R}$, $R_r f := \int_a^b e^{-irt} f(t) dt$ is uniformly bounded and by the lemma of Riemann-Lebesgue we obtain $R_r g \rightarrow 0$ for $|r| \rightarrow \infty$ and every step function $g(\cdot)$. Therefore, by [45], III.4.5, $R_r f \rightarrow 0$ for $|r| \rightarrow \infty$ and every $f \in R([a, b], E)$.

□

Lemma A.2

Let $f(\cdot)$ be a differentiable function from $\mathbb{R}^+$ into E such that

$$\int_0^\infty ||f(s)|| \, ds \quad (= \lim_{t \rightarrow \infty} \int_0^t ||f(s)|| \, ds)$$

exists. Define a continuous function $g(\cdot)$ from $\mathbb{R}$ into E by

$$g(s) := \int_0^\infty e^{-ist} f(t) \, dt .$$

Then, for every $t > 0$,

$$(\#) \quad f(t) = \lim_{r \rightarrow \infty} \frac{1}{2\pi} \int_{-r}^r e^{its} g(s) \, ds .$$

Proof:

a) The functions $g_T(\cdot)$, $g_T(s) := \int_0^T e^{-ist} f(t) \, dt$, converge uniformly on $\mathbb{R}$ to $g(\cdot)$. Using the equations

$$\begin{aligned} \int_{-r}^r e^{its} g(s) \, ds &= \lim_{T \rightarrow \infty} \int_{-r}^r e^{its} g_T(s) \, ds \\ &= \lim_{T \rightarrow \infty} \int_{-r}^r \int_0^T e^{is(t-u)} f(u) \, du \, ds \\ &= \lim_{T \rightarrow \infty} \int_0^T f(u) \int_{-r}^r e^{is(t-u)} \, ds \, du \\ &= 2 \int_0^\infty \frac{1}{t-u} \sin[r(t-u)] f(u) \, du , \end{aligned}$$

equation $(\#)$ becomes

$$(\#\#) \quad f(t) = \lim_{R \rightarrow \infty} \frac{1}{\pi} \int_0^R \frac{1}{t-u} \sin[r(t-u)] f(u) du .$$

b) Choose a constant $R > \max\{2t, t+1\}$ such that

$$\frac{1}{\pi} \int_R^\infty ||f(t)|| dt < \varepsilon/3 . \text{ Then, for every } r \in \mathbb{R} :$$

$$|| \frac{1}{\pi} \int_R^\infty \frac{1}{t-u} \sin[r(t-u)] f(u) du || < \varepsilon/3 . \text{ From the}$$

continuity of the functions $s \rightarrow \frac{1}{s} f(t-s)$ on $[t-R, -t]$ and the lemma of Riemann-Lebesgue it follows that there is a $r_0 > 0$ such that for every $r \in \mathbb{R}$ with $|r| > r_0$:

$$\begin{aligned} & || \frac{1}{\pi} \int_{2t}^R \frac{1}{t-u} \sin[r(t-u)] f(u) du || \\ &= || \frac{1}{\pi} \int_{t-R}^{-t} \sin rs \frac{1}{s} f(t-s) ds || < \varepsilon/3 . \end{aligned}$$

Therefore, for every $\varepsilon > 0$, there is a $r_0 > 0$ such that, for every $r \in \mathbb{R}$ with $|r| > r_0$, we have

$$\begin{aligned} & || f(t) - \frac{1}{\pi} \int_0^\infty \frac{1}{t-u} \sin[r(t-u)] f(u) du || \\ &\leq || f(t) - \frac{1}{\pi} \int_0^{2t} \frac{1}{t-u} \sin[r(t-u)] f(u) du || + 2\varepsilon/3 . \end{aligned}$$

The only thing left to prove is the existence of a positive real number $r_1 \geq r_0$ such that, for every $r \in \mathbb{R}$ with $|r| > r_1$ we have

$$|| f(t) - \frac{1}{\pi} \int_0^{2t} \frac{1}{t-u} \sin[r(t-u)] f(u) du || < \varepsilon/3 .$$

c) By

$$\begin{aligned}
 \int_0^{2t} \frac{1}{t-u} \sin[r(t-u)] f(u) du &= \int_{-t}^t \sin rs \frac{1}{s} f(t-s) ds \\
 &= \int_0^t \sin rs \frac{1}{s} f(t-s) ds + \int_{-t}^0 \sin rs \frac{1}{s} f(t-s) ds \\
 &= \int_0^t \sin rs \frac{1}{s} f(t-s) ds + \int_0^t \sin rs \frac{1}{s} f(t+s) ds \\
 &= \int_0^t \sin rs \frac{1}{s} (f(t-s) + f(t+s)) ds \quad \text{and}
 \end{aligned}$$

$\lim_{r \rightarrow \infty} \frac{1}{\pi} \int_0^t \frac{1}{s} \sin rs ds = 1/2$ (see [12], p.165) we obtain the following statements:

$$\lim_{r \rightarrow \infty} \frac{1}{\pi} \int_0^{2t} \frac{1}{t-u} \sin[r(t-u)] f(u) du = f(t)$$

if and only if

$$\begin{aligned}
 \lim_{r \rightarrow \infty} \frac{1}{\pi} \int_0^t \sin rs \frac{1}{s} (f(t-s) + f(t+s)) ds \\
 = \lim_{r \rightarrow \infty} \frac{1}{\pi} \int_0^t \frac{1}{s} \sin rs (2f(t)) ds
 \end{aligned}$$

if and only if

$$\lim_{r \rightarrow \infty} \frac{1}{\pi} \int_0^t \sin rs \left[\frac{1}{s} (f(t+s) - f(t)) + \frac{1}{s} (f(t+s) - f(t)) \right] ds = 0.$$

By the lemma of Riemann-Lebesgue and the continuity of the functions $s \rightarrow \frac{1}{s} (f(t+s) - f(t))$ and $s \rightarrow \frac{1}{s} (f(t+s) - f(t))$ on $[0, t]$ we see that the last statement is true. There-

fore there is a $r_1 \geq r_0$ such that

$$|| f(t) - \frac{1}{\pi} \int_0^{2t} \frac{1}{t-u} \sin[r(t-u)] f(u) du || < \varepsilon/3$$

for every $r \in \mathbb{R}$ with $|r| > r_1$.

□

Proposition A.3

(The inversion formula for the Laplace transform) Let f be a differentiable function from $\mathbb{R}^+$ into a Banach space E such that

$$\int_0^\infty ||e^{-\mu t} f(t)|| dt \text{ exists for some } \mu \in \mathbb{C}.$$

Then, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq \operatorname{Re} \mu$, the Laplace transform $L(f)(\cdot)$ of $f(\cdot)$ exists in λ and, for $t > 0$ and every $p > \operatorname{Re} \mu$:

$$\begin{aligned} f(t) &= e^{pt} \lim_{r \rightarrow \infty} \frac{1}{2\pi} \int_{-r}^r e^{ist} L(f)(p+is) ds \\ &= \frac{1}{2\pi i} \int_{p-i\infty}^{p+i\infty} e^{\lambda t} L(f)(\lambda) d\lambda. \end{aligned}$$

Proof:

If $\int_0^\infty ||e^{-\mu t} f(t)|| dt$ exists for some $\mu \in \mathbb{C}$ then it can be seen by

$$\begin{aligned} \int_a^b ||e^{-\lambda t} f(t)|| dt &= \int_a^b e^{-\operatorname{Re}(\lambda-\mu)t} ||e^{-\mu t} f(t)|| dt \\ &\leq \int_a^b ||e^{-\mu t} f(t)|| dt \end{aligned}$$

that $\int_0^\infty ||e^{-\lambda t} f(t)|| dt$ and hence

$$L(f)(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt \text{ exists}$$

for $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq \operatorname{Re} \mu$.

Let $p > \operatorname{Re} \mu$. Then $\int_0^\infty ||e^{-pt} f(t)|| dt$ exists. Defining a function $g(\cdot)$ on $\mathbb{R}$ by

$$\begin{aligned} g(s) &:= \int_0^t e^{-ist} e^{-pt} f(t) dt = \int_0^\infty e^{-(p+is)t} f(t) dt \\ &= L(f)(p+is) \end{aligned}$$

we obtain by lemma A.2 :

$$e^{-pt} f(t) = \lim_{r \rightarrow \infty} \frac{1}{2\pi} \int_{-r}^r e^{ist} g(s) ds$$

$$\begin{aligned} \text{or } f(t) &= e^{pt} \lim_{r \rightarrow \infty} \frac{1}{2\pi} \int_{-r}^r e^{ist} L(f)(p+is) ds \\ &= \frac{1}{2\pi i} \int_{p-i\infty}^{p+i\infty} e^{\lambda t} L(f)(\lambda) d\lambda . \end{aligned}$$

□

Before applying proposition A.3 to semigroup theory we will recall some abscissas of the Laplace transform of a strongly continuous semigroup. Let A be the generator of a strongly continuous semigroup $(T(t))$ and let $x \in E$. The abscissa of absolute convergence of the Laplace transform of $t \rightarrow T(t)x$ coincides with the exponential growth bound, i.e.

$$w(x) := \inf \{ \operatorname{Re} \lambda : \int_0^\infty \| e^{-\lambda t} T(t)x \| dt \text{ exists} \}$$

$$= \inf \{ w : \| T(t)x \| \leq M_w e^{wt} \text{ for every } t \geq 0 \}$$

$$\text{and } w_0 := \inf \{ \operatorname{Re} \lambda : \int_0^\infty \| e^{-\lambda t} T(t)x \| dt \text{ exists for every } x \in E \}$$

$$= \inf \{ w : \| T(t) \| \leq M_w e^{wt} \text{ for every } t \geq 0 \}.$$

For a proof see [27]. The abscissa of simple convergence of the Laplace transform of $t \rightarrow T(t)x$ coincides with the exponential growth bound of the solutions of the Cauchy problem $u'(t) = Au(t)$, i.e.

$$w_1 := \inf \{ \operatorname{Re} \lambda : \int_0^\infty e^{-\lambda t} T(t)x dt \text{ exists for every } x \in E \}$$

$$= \inf \{ w : \| T(t)x \| \leq M_{w,x} e^{wt} \text{ for every } x \in D(A) \text{ and every } t \geq 0 \}.$$

For the proof and other equations for w_1 see [37]. The constant

$$\gamma(x) := \inf \{ w : \lambda \rightarrow R(\lambda, A)x \text{ has an holomorphic extension in the halfplane } \{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda > w \} \}$$

is called the abscissa of holomorphy of the Laplace transform of $t \rightarrow T(t)x$ and we will denote the extension of $R(\cdot, A)x$ by $R_e(\cdot, A)x$. The spectral bound $s(A) := \inf \{ \operatorname{Re} \lambda : \lambda \in \sigma(A) \}$ coincides with

$\sup \{ \gamma(x) : x \in E \}$ and the following inequalities hold:

$$s(A) \leq w_1 \leq w_0.$$

We emphasize that there are semigroups with $s(A) < w_1 < w_0$ (for an example, see [37]). For a positive semigroup we have always $s(A) = w_1$ whereas the equality $s(A) = w_0$ may not hold in general (see [37] and [22]). Moreover, for some important classes of semigroups the equalities $s(A) = w_1 = w_0$ are true. For example there is equality if A is the generator of a semigroup which is norm continuous for $t > t_0 \geq 0$ (compact or differentiable or holomorphic semigroups, see [8]) or if A is the generator of a positive semigroup on either (a) $E = L^1(\mu)$ ([10]), (b) $E = L^2(\mu)$ ([23]), (c) E a unital C^* -algebra ([24]), (d) E with an order unit or α -directed (for example $C_0(\mathbb{R})$, see [5]). For further results on the connection between $s(A)$ and the growth of the solutions of (ACP₁) see, for example, [32], [33], [38] and [51]. If A is the generator of a semigroup which is norm continuous for $t > t_0 \geq 0$ then there exists a spectral value λ with $\operatorname{Re} \lambda = s(A)$ (see the proposition below) and if A is the generator of a positive semigroup then we have even $s(A) \in \sigma(A)$ (see [22]). Therefore, in many cases the growth bound w_1 which plays a central role in theorem A.5 (the main result of the appendix) is determined by a "maximal" spectral value of the generator A .

Proposition A.4

Let A be the generator of a semigroup which is norm continuous for $t > t_0 \geq 0$. Then there is a $\lambda \in \sigma(A)$ with $\operatorname{Re} \lambda = s(A) = w_1 = w_0$.

Proof:

For every eventually norm continuous semigroup we have $s(A) = w_1 = w_0$ (see [8]). Therefore we have to prove the existence of a spectral value λ with $\operatorname{Re} \lambda = s(A)$. For every bounded operator T the following is true:

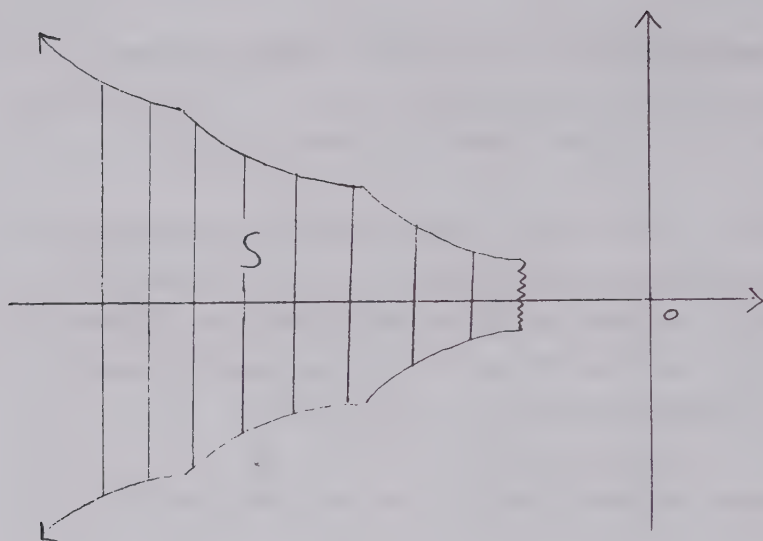
$$r(T)^n = r(T^n) \leq \|T^n\| \quad \text{and} \quad r(T) = \lim \|T^n\|^{1/n}.$$

Choose $a > w_1$. Then

$$\sum_{n=0}^{\infty} (-1)^n [\lambda - (a+ir)]^n R(a+ir, A)^{n+1}$$

exists for all $\lambda \in \{s+ir : |s-a| < \frac{1}{2} \|R(a+ir, A)^n\|^{-1/n}\}$ and coincides with $R(\lambda, A)$.

Therefore, if we can prove that for every $\varepsilon > 0$ there exists a natural number n and a positive number r_0 such that for every $r \in \mathbb{R}$ with $|r| \geq r_0$ we have $\|R(a+ir, A)^n\|^{1/n} < \varepsilon$, then the spectrum of A is contained in a subset S of $\mathbb{C}$ of the following type:



More precisely: there exists a continuous, monotone function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\lim_{x \rightarrow \infty} f(x) = +\infty$ such that $\rho(A) \supset \{ \lambda : \lambda = a+ib, |b| \geq f(|a|) \}$.

- (a) Choose $n \in \mathbb{N}$ such that for every $x \in E$ and every $r \in \mathbb{R}$:

$$|| \int_0^t \frac{1}{n!} t^n e^{-(a+ir)t} T(t)x dt || < \frac{1}{3} \epsilon^{n+1}.$$

- (b) Choose $T > 0$ such that for every $x \in E$ and every $r \in \mathbb{R}$:

$$|| \int_T^\infty \frac{1}{n!} t^n e^{-(a+ir)t} T(t)x dt || < \frac{1}{3} \epsilon^{n+1}.$$

- (c) Choose $r_0 > 0$ such that for every $r \in \mathbb{R}$ with $|r| > r_0$:

$$|| \int_t^T e^{-irt} \frac{1}{n!} t^n e^{-at} T(t)x dt || < \frac{1}{3} \epsilon^{n+1}$$

(Lemma A.1).

Since $R(a+ir, A)^{n+1}x = \int_0^\infty \frac{1}{n!} t^n e^{-(a+ir)t} T(t)x dt$ we get for the $n \in \mathbb{N}$ choosen in (a) :

$$||R(a+ir, A)^{n+1}||^{1/n+1} < \varepsilon \quad \text{for every } r \in \mathbb{R} \text{ with } |r| > r_0, \quad r_0 \text{ as in (b).}$$

□

The main result of the appendix is a version of the theorem of the inversion of the Laplace transform which is stronger than the one given in the book of E.Hille and R.S. Phillips [27] since the condition $p > \max\{0, w_0\}$ is replaced by the weaker condition $p > w_1$. By the previous remarks we have seen that in many cases the constant w_1 is determined by a "maximal" spectral value of A . The proof follows immediatly from lemma A.3 and the remarks on the growth bounds $w(x)$ and w_1 .

Theorem A.5

(The inversion formula of

the Laplace transform of a strongly continuous semigroup)

Let A be the generator of a strongly continuous semigroup $(T(t))$ on a Banach space E . The following statements are true:

(i) If $x \in D(A)$ and $p > w(x)$, then

$$T(t)x = \frac{1}{2\pi i} \int_{p-i\infty}^{p+i\infty} e^{\lambda t} R_e(\lambda, A)x d\lambda.$$

(ii) If $p > w_1$, then

$$T(t)x = \frac{1}{2\pi i} \int_{p-i\infty}^{p+i\infty} e^{\lambda t} R(\lambda, A)x \, d\lambda .$$

for every $x \in D(A)$.

□ □ □

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LEBENS LAUF

Am 18.10.54 wurde ich, Frank Michael Neubrandner, als Sohn des Diplom-Kaufmanns Eberhard Neubrandner und seiner Ehefrau Elisabeth, geb. Walz, in Reutlingen geboren. Von 1960 bis 1964 besuchte ich die Matthäus-Beger-Schule in Reutlingen, von der ich auf das Friedrich-List-Gymnasium in Reutlingen wechselte. Dort legte ich 1973 die Reifeprüfung ab.

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Seit dem Wintersemester 1976/77 bin ich am Mathematischen Institut der Universität Tübingen als wissenschaftliche Hilfskraft tätig.

Im November 1981 nahm ich im Rahmen eines DAAD-Stipendiums an einem 'post-graduate course' Functional Analysis in Dubrovnik teil. Im Sommer 1983 ermöglichte mir ein DAAD-Dissertationsstipendium einen halbjährigen Studienaufenthalt an der Tulane University, New Orleans.

Meine akademischen Lehrer im Fach Mathematik waren die Herren Kaup, Nagel, Salzmann, Schaefer, Schlotterbeck, Siebert und Volger, im Fach Wirtschaftswissenschaften die Herren Schaich, Scheele und Strecker.

